Building symplectic manifolds using hyperbolic geometry

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Abstract

This is a survey of the symplectic part of [7]. It is known that a hyperbolic manifold of even dimension is the base of a bundle whose total space admits a natural symplectic form ([11, 3]). We use this together with a construction resembling that of the Kummer surface to produce a simply-connected symplectic 6-manifold with vanishing first Chern class but no compatible complex structure. The role of the manifold-with-involution—the complex torus in Kummer's original construction—is in our case played by a beautiful hyperbolic 4-manifold discovered by Davis [4].

1 Introduction

This article gives a survey of some of the ideas of [7]. The goal is to describe the construction of a simply-connected symplectic 6-manifold which has vanishing first Chern class, but admits no compatible integrable complex structure. This is, to the best of our knowledge, the first such example to appear in the literature. The construction combines hyperbolic geometry and a resolution-of-singularities argument reminiscent of the Kummer surface. (The article [7] also gives an analogous construction of simply-connected complex threefolds with trivial canonical bundle, but which admit no symplectic forms. We do not describe this here.)

It follows from the theory of coadjoint orbits that every hyperbolic $2n$-manifold M is the base of a fibration $X \to M$ whose total space is a symplectic manifold—see §2.1. This can also be seen as a special case of the work of Reznikov [11] concerning symplectic structures on twistor spaces. There, Reznikov considers more general metrics satisfying a curvature inequality. Since we need only the hyperbolic case, we do not describe this here. (For more information on this see also the discussion in [8], which focuses on this inequality in the 4-dimensional case and the related work of Davidov–Muškarov–Grantcharov [3].)

For $n \geq 3$, these symplectic manifolds have c_1 a positive multiple of the symplectic class—the symplectic analogues of Fano varieties, if you will.

When $n = 2$, the first Chern class vanishes, giving symplectic "Calabi–Yaus".

The fundamental group of a symplectic manifold built this way is equal to the fundamental group of the hyperbolic base and so this does not give simply-connected compact examples directly. To get around this we use a construction analogous to that of the Kummer surface. We use a special hyperbolic 4-manifold, called the Davis manifold, to build a symplectic 6-manifold with an involution. Dividing by the involution kills the fundamental group and leaves a symplectic orbifold with vanishing first Chern class. We then resolve the singularities to give a smooth simply-connected symplectic manifold with $c_1 = 0$. This manifold has $b_3 = 0$ and so cannot support a Kähler structure with $c_1 = 0$. (Simply-connected Kähler manifolds with $c_1 = 0$ in fact have holomorphically trivial canonical bundle, hence $b_3 \geq 2$ because of the existence of a holomorphic volume form.)

2 From hyperbolic to symplectic manifolds

2.1 A hyperbolic coadjoint orbit

Given a Lie group G and a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$, it is a standard fact that $\mathcal{O}$ carries a natural symplectic structure. Given $u, v \in \mathfrak{g}$ and $a \in \mathcal{O}$, the symplectic form at a evaluated on the vectors in $T_a\mathcal{O}$ corresponding to u, v is simply $a([u, v])$. We will apply this general fact to the group of isometries of hyperbolic $2n$-dimensional space.

The group of orientation-preserving isometries of hyperbolic space H^m can be identified with $\mathrm{SO}(m, 1)$, the orientation-preserving linear isomorphisms of $\mathbb{R}^{m+1}$ which also preserve the Minkowski metric $x_0^2 - x_1^2 - \cdots - x_m^2$. The Lie algebra $\mathfrak{so}(m, 1)$ is the space of $(m + 1) \times (m + 1)$ matrices of the form

$$\begin{pmatrix} 0 & u^t \\ u & A \end{pmatrix} \tag{1}$$

where u is a column vector in $\mathbb{R}^m$ and $A \in \mathfrak{so}(m)$. Those elements with $u = 0$ generate $\mathfrak{so}(m) \subset \mathfrak{so}(m, 1)$, which correspond to rotations of H^m about a fixed point. The Killing form is definite on $\mathfrak{so}(m, 1)$ and so gives an equivariant isomorphism $\mathfrak{so}(m, 1) \cong \mathfrak{so}(m, 1)^*$. It follows that adjoint and coadjoint orbits can be identified.

We now restrict to the case when $m = 2n$ is *even*. Consider the adjoint orbit of

$$\xi = \begin{pmatrix} 0 & 0 \\ 0 & J_0 \end{pmatrix}$$

where $J_0 \in \mathfrak{so}(2n)$ is a choice of almost complex structure on $\mathbb{R}^{2n}$ (i.e., $J_0^2 = -1$). The stabiliser of ξ is $\mathrm{U}(n)$. We write Z for the corresponding orbit (or Z_{2n} when we need to emphasise the value of n).

Thought of as a coadjoint orbit, Z is symplectic. The identification $Z_{2n} \cong \mathrm{SO}(2n,1)/\mathrm{U}(n)$ shows $\dim Z_{2n} = n(n+1)$. It fibres naturally over hyperbolic space $H^{2n} \cong \mathrm{SO}(2n,1)/\mathrm{SO}(2n)$ with fibre isomorphic to the symmetric space $\mathrm{SO}(2n)/\mathrm{U}(n)$. This space can be identified with the collection of linear orthogonal complex structures on $\mathbb{R}^{2n}$ inducing a fixed orientation—that is to say orthogonal endomorphisms J of $\mathbb{R}^{2n}$ with $J^2 = -1$ and such that a basis of the form $v_1, Jv_1, \ldots, v_n, Jv_n$ gives a fixed orientation. It follows that Z is the twistor space of H^{2n}, although we don't directly use twistor methods here.

We next find an almost complex structure on Z which is compatible with its symplectic structure. To do this we will use an equivariant description of the tangent bundle.

Lemma 1. *There is an isomorphism of* $\mathrm{U}(n)$*-representation spaces:*

$$\mathfrak{so}(2n,1) \cong \mathfrak{u}(n) \oplus \Lambda^2(\mathbb{C}^n)^* \oplus \mathbb{C}^n.$$

Given a point $z \in Z$ *with stabiliser* $\mathrm{U}(n) \subset \mathrm{SO}(2n,1)$ *there is a* $U(n)$*-equivariant isomorphism*

$$T_z Z \cong \Lambda^2(\mathbb{C}^n)^* \oplus \mathbb{C}^n, \tag{2}$$

in which $\Lambda^2(\mathbb{C}^n)^*$ *is tangent to the fibre of the projection* $Z \to H^{2n}$*.*

Under this isomorphism, the symplectic form on $T_z Z$ *is a positive multiple of the standard form on* $\Lambda^2(\mathbb{C}^n)^* \oplus \mathbb{C}^n$ *induced by the Euclidean structure on* $\mathbb{C}^n$*.*

Proof. There is a $\mathrm{U}(n)$-equivariant isomorphism $\mathfrak{so}(2n) \cong \mathfrak{u}(n) \oplus \Lambda^2(\mathbb{C}^n)^*$. To see this, write $\mathfrak{so}(2n) \cong \Lambda^2(\mathbb{R}^{2n})^*$. Given a choice of almost complex structure on $\mathbb{R}^{2n}$, any real 2-form a can be written uniquely as $a = \alpha + \beta + \bar{\beta}$ where $\alpha \in \Lambda^{1,1}_{\mathbb{R}}$ is a real $(1,1)$-form and $\beta \in \Lambda^{2,0}$. Identifying a with (α, β) gives a $\mathrm{U}(n)$-equivariant decomposition $\Lambda^2_{\mathbb{R}} \cong \Lambda^{1,1}_{\mathbb{R}} \oplus \Lambda^{2,0}$. Now, via the metric, $\Lambda^{1,1}_{\mathbb{R}}$ is identified with the skew-Hermitian matrices $\mathfrak{u}(n)$ and this gives the claimed isomorphism.

There is also an $\mathrm{SO}(2n)$-equivariant isomorphism $\mathfrak{so}(2n,1) \cong \mathfrak{so}(2n) \oplus \mathbb{R}^{2n}$. In the above form (1) the $\mathfrak{so}(2n)$ summand is given by $u = 0$ whilst the $\mathbb{R}^{2n}$ summand by $A = 0$. Combining these two completes the proof of the isomorphism (2).

Next we take care of the symplectic form. By $\mathrm{U}(n)$-equivariance, the form on $T_z Z$ must be proportional under (2) to the form induced by the Euclidean structure on $\mathbb{C}^n$. To show the constant of proportionality is positive, first check that the forms are genuinely equal in the case $n = 1$, where $\mathfrak{so}(2,1) \cong \mathfrak{u}(1) \oplus \mathbb{C}$ as a $\mathrm{U}(1)$-representation. This amounts to the fact that $Z_2 = H^2$ with symplectic form the hyperbolic area form. Next, use induction and the fact that the decompositions of $\mathfrak{so}(2n,1)$ and $\mathfrak{so}(2n+2,1)$ from above are compatible with the obvious inclusions of the summands induced by a choice of $\mathbb{C}^n \subset \mathbb{C}^{n+1}$. $\square$

We define an $\mathrm{SO}(2n,1)$-invariant almost complex structure J on Z defined by declaring (2) to be a complex linear isomorphism. Notice that, by Lemma 1, J is compatible with the symplectic structure on Z.

With respect to J the tangent bundle splits $TZ = V \oplus H$ as a sum of complex vector bundles with $V \cong \Lambda^2 H^*$, corresponding to (2). The sub-bundle V is those vectors tangent to the fibres of $Z \to H^{2n}$. We remark in passing that in the twistorial picture, this splitting is simply the decomposition of TZ induced by the Levi–Civita connection of H^{2n}. From this point of view, the almost complex structure J is the (non-integrable) "Eells–Salamon" structure [5], given by reversing the (integrable) "Atiyah–Hitchin–Singer" structure [1] in the vertical directions.

Lemma 2. $c_1(Z_{2n}) = (2-n)c_1(H)$.

Proof. This follows from $TZ = \Lambda^2 H^* \oplus H$ along with the fact that for any complex rank n vector bundle E, $c_1(\Lambda^2 E) = (n-1)c_1(E)$. $\square$

We next determine the symplectic class of Z. First, consider the restriction of the symplectic structure to the fibres of $Z \to H^{2n}$. It follows from (2) that the fibres are symplectic and almost-complex submanifolds. Moreover, the stabiliser $\mathrm{SO}(2n) \subset \mathrm{SO}(2n,1)$ of a point $x \in H^{2n}$ acts on the fibre F_x over x preserving both these structures. As mentioned above, $F_x \cong \mathrm{SO}(2n)/\mathrm{U}(n)$ is a symmetric space (the space of linear orthogonal complex structures on $T_x H^{2n}$ inducing a fixed orientation). The standard theory of symmetric spaces gives F_x a symmetric Kähler structure. It follows from $\mathrm{SO}(2n)$-equivariance that this must agree with the restriction of the symplectic and almost complex structures from Z.

It is also standard that the symmetric Kähler structure on $\mathrm{SO}(2n)/\mathrm{U}(n)$ comes from a projective embedding. In this particular case, we can describe this as follows. Let $E \to F$ denote the "tautological" bundle over $F = \mathrm{SO}(2n)/\mathrm{U}(n)$. I.e., each point of F is a complex structure on $\mathbb{R}^{2n}$; the fibre of E at a point $j \in F$ is the complex vector space $(\mathbb{R}^{2n}, j)$. The bundle $\det E^*$ is ample and $c_1(\det E^*) = -c_1(E)$ is represented by the symmetric symplectic form on F.

In our situation, the splitting (2) tells us that the tautological bundle of the fibre F_x is simply $H|_{F_x}$. It follows that on restriction to a fibre, the symplectic class agrees with $-c_1(H)$. However, topologically, $Z \cong F \times H^{2n}$ is homotopic to F. Since their fibrewise restrictions agree, it follows that $-c_1(H)$ is equal to the symplectic class of Z. Hence, writing ω for the symplectic form on Z, we have:

Proposition 3. $c_1(Z_{2n}) = (n-2)[\omega]$.

2.2 Compact quotients

The symplectic form and almost-complex structure on Z as well as the splitting $TZ = V \oplus H$ and identification $V \cong \Lambda^2 H^*$ are $\mathrm{SO}(2n,1)$-invariant. It follows that all these arguments, in particular Proposition 3, apply equally to smooth quotients of Z_{2n} by subgroups of $\mathrm{SO}(2n,1)$. Let $\Gamma \subset \mathrm{SO}(2n,1)$ be the fundamental group of a compact hyperbolic $2n$-manifold M. Γ acts by symplectomorphisms on Z_{2n} to give as quotient a symplectic manifold of dimension $n(n+1)$. It fibres $Z_{2n}/\Gamma \to M$ as the twistor space of M. In this way we can use compact hyperbolic manifolds to produce compact symplectic manifolds.

When $n = 1$, $Z_2/\Gamma = M$, the symplectic form is the hyperbolic area form and we have simply the recovered the hyperbolic surface itself.

When $n \geq 2$ the situation is more interesting. For $n = 2$, the fibre $\mathrm{SO}(4)/\mathrm{U}(2)$ is the 2-sphere. So the quotient Z_4/Γ is a 6-manifold which is the total space of a 2-sphere bundle over M. Moreover, Proposition 3 says $c_1(Z_4/\Gamma) = 0$ and so hyperbolic 4-manifolds lead to symplectic "Calabi–Yau" manifolds. This was seemingly first observed in [8]. That article also describes a more general approach for constructing symplectic forms on the total space of S^2-bundles over 4-manifolds, which involves certain $\mathrm{SO}(3)$-connections, called *definite connections*. From that point of view, the Levi–Civita connection on $\Lambda^+ \to H^4$ is an example of a definite connection.

Remark 4. It will be important in what follows to note that when $n = 2$, not only do the symplectic manifolds have $c_1 = 0$, but they actually have a *preferred section of the almost canonical bundle.* This is because of the isomorphism (2) which in the case of $n = 2$ implies

$$\Lambda^3 T^*_z Z \cong \Lambda^2(\mathbb{C}^2)^* \otimes \Lambda^2(\mathbb{C}^2)$$

Thus the almost canonical bundle over Z is not just trivial, but trivial in a natural $\mathrm{SO}(4,1)$-invariant way. It follows that the almost canonical bundle of the compact quotients also have natural trivialisations.

When $n \geq 3$, Proposition 3 says that compact hyperbolic $2n$-manifolds give compact symplectic manifolds for which c_1 is a positive multiple of $[\omega]$ (for $n \geq 3$). These might be considered as symplectic analogues of Fano varieties in algebraic geometry. Recall that $\dim Z_{2n} = n(n+1)$ so the lowest dimension of "symplectic Fano" which can be achieved in this way is 12. In this case, the fibration over M^6 has fibres $\mathrm{SO}(6)/\mathrm{U}(3) \cong \mathbb{CP}^3$.

We remark that, for $n \geq 2$, no compact manifold produced in this way can be Kähler. This is because, for $m > 2$ no discrete co-compact lattice in $\mathrm{SO}(m,1)$ can arise as the fundamental group of a compact Kähler manifold. In particular, the "Fanos" arising from hyperbolic manifolds of dimension at least 6 are non-Kähler. These examples (originally appearing in Reznikov's article [11]) are, to the best of our knowledge, the first non-Kähler symplectic "Fano" manifolds.

2.3 A Kähler description

In what follows, it will be important to have an alternative description of Z in the case of H^4. In fact, Z has an *integrable* complex structure which is compatible with the symplectic form. This complex structure is *not* $\mathrm{SO}(4,1)$-invariant (otherwise it would descend to all quotients) but is only invariant under the smaller group $\mathrm{SO}(4)$.

This can be described as follows. We denote by Y the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{CP}^1$; write $\pi\colon Y \to \mathbb{CP}^1$ for the bundle projection. There is a map $p\colon Y \to \mathbb{C}^2 \oplus \mathbb{C}^2 \cong \mathbb{C}^4$ given by adding the maps $\mathcal{O}(-1) \to \mathbb{C}^2$. We can define a Kähler metric on Y by pulling back the metrics from $\mathbb{CP}^1$ and $\mathbb{C}^4$:

$$\omega = \pi^* \omega_{\mathrm{FS}} + p^* \omega_{\mathbb{C}^4}.$$

The points (x, y) and (z, w) of $\mathbb{C}^2$ span the same line if and only if $xw - yz = 0$. It follows that the image of p is the quadric cone $Q = \{xw - yz = 0\}$. The map $p\colon Y \to Q$ is called the small resolution of Q.

(Strictly speaking, there are two small resolutions, depending on whether one identifies $\mathbb{C}^2 \oplus \mathbb{C}^2 \cong \mathbb{C}^4$ by sending $((x, y), (z, w))$ to (x, y, z, w) or to (x, z, y, w). This latter gives another map $p'\colon Y \to Q$. There is no biholomorphism of Y which swaps p and p', so they are genuinely inequivalent resolutions. For more details see, for example, the article [12] which gives a more thorough description of the small resolutions.)

The group $\mathrm{SO}(4, \mathbb{C})$ of complex linear isomorphisms of $\mathbb{C}^4$ with determinant 1 and preserving the quadratic form $xw - zy$ acts on Q. This action lifts to a holomorphic action on the resolution. If we want to also preserve the Kähler form, however, we must restrict to $\mathrm{SO}(4, \mathbb{C}) \cap \mathrm{U}(4)$. The Hermitian and complex forms determine a conjugation map on $\mathbb{C}^4$, the real subspace fixed by the conjugation is the copy of $\mathbb{R}^4 \subset \mathbb{C}^4$ where the two forms agree. From this it follows that $\mathrm{SO}(4, \mathbb{C}) \cap \mathrm{U}(4) \cong \mathrm{SO}(4)$. Here $\mathrm{SO}(4)$ acts on $\mathbb{C}^4 \cong \mathbb{R}^4 \oplus i\mathbb{R}^4$ by extending the standard action on $\mathbb{R}^4$ by complex linearity. (If we choose coordinates in which the complex quadratic form is given by $z_1^2 + \cdots + z_4^2$, the decomposition $\mathbb{C}^4 \cong \mathbb{R}^4 \oplus i\mathbb{R}^4$ just corresponds to $z = \mathrm{Re}\, z + i \,\mathrm{Im}\, z$.)

Projecting onto the real subspace gives a surjection $t\colon Q \to \mathbb{R}^4$. Away from the origin the fibres are two-spheres. The map lifts to the small resolution where it becomes a genuine S^2-bundle $t\colon Y \to \mathbb{R}^4$. This map is $\mathrm{SO}(4)$-equivariant and so we can see Y in a similar fashion to $Z \to H^4$. Given this similarity the following is not surprising.

Proposition 5. *There is an* $\mathrm{SO}(4)$*-equivariant symplectomorphism between* Y *and* Z*.*

This was proved first in [8] and then in a different way in [7]. We omit the details here.

3 A simply-connected symplectic Calabi–Yau

For the rest of this survey we focus on the construction of a simply-connected non-Kähler symplectic Calabi–Yau 6-manifold, which appears in [7]. The discussion above shows that a hyperbolic 4-manifold carries a 2-sphere bundle whose total space is naturally a symplectic "Calabi–Yau" manifold, with $c_1 = 0$. It is not possible to apply this directly to obtain a simply-connected compact example because, of course, compact hyperbolic manifolds have large fundamental group. To get around this problem we use a tactic resembling Kummer's construction of a K3 surface.

3.1 The Kummer construction

We begin by briefly recalling Kummer's construction. Let $A \cong \mathbb{C}^2/\mathbb{Z}^4$ be a complex torus. The involution $z \mapsto -z$ descends to A where it fixes 16 points. The resulting quotient $A/\mathbb{Z}_2$ is singular, but simply-connected as a topological space (that is to say, we do not consider the orbifold fundamental group here). Moreover, the holomorphic volume form $\mathrm{d}z_1 \wedge \mathrm{d}z_2$ on A is $\mathbb{Z}_2$-invariant and so $A/\mathbb{Z}_2$ has holomorphically trivial canonical bundle. To produce a smooth simply-connected surface with trivial canonical bundle, we now resolve the 16 singularities.

To give a local model for the resolution one can start with the model for the blow-up of a point $\mathcal{O}(-1) \to \mathbb{C}^2$. The involution $z \mapsto -z$ of $\mathbb{C}^2$ lifts to $\mathcal{O}(-1)$ where it now fixes the whole exceptional curve. The singularity of $\mathcal{O}(-1)/\mathbb{Z}_2$ is resolved by squaring the normal bundle to the exceptional curve. In other words, taking the "square-root" gives a map $\mathcal{O}(-2) \to \mathcal{O}(-1)/\mathbb{Z}_2$. Now composing with projection to $\mathbb{C}^2/\mathbb{Z}_2$ gives a resolution $\mathcal{O}(-2) \to \mathbb{C}^2/\mathbb{Z}_2$ in which the singularity has been replaced by a curve of self-intersection -2. Doing this locally at each of the 16 singular points of $A/\mathbb{Z}_2$ we get a smooth resolution $X \to A/\mathbb{Z}_2$. Moreover it is straightforward to verify that, since we have changed the topology only in codimension 2, the fundamental group is unaffected and so $\pi_1(X) = 1$.

An important feature of the model resolution $p\colon \mathcal{O}(-2) \to \mathbb{C}^2/\mathbb{Z}_2$ is that the pull-back $p^*(\mathrm{d}z_1 \wedge \mathrm{d}z_2)$, defined a priori away from the exceptional locus, actually extends to a global nowhere-vanishing holomorphic volume form on $\mathcal{O}(-2)$. This is described by saying the resolution is *crepant*. It follows that, when thought of as a form on $A/\mathbb{Z}_2$ (away from the singularities), $\mathrm{d}z_1 \wedge \mathrm{d}z_2$ pulls back to X and extends to a holomorphic trivialisation of the canonical bundle. In other words, X is a simply-connected complex surface with trivial canonical bundle.

3.2 The Davis manifold

In order to implement a similar construction in our situation, we first need a hyperbolic 4-manifold which can play the role of A. For this we use a beautiful manifold found by Davis [4]. (See also the article of Ratcliffe–Tschantz [10] where the homology of the Davis manifold is computed.)

The Davis manifold M is built using a regular polytope called the 120-cell (or hecatonicosachoron). The 120-cell is a four-dimensional regular solid with 120 three-dimensional faces—the "cells"—each of which is a solid dodecahedron. Each edge is shared by 3 dodecahedra and each vertex by 4 dodecahedra. In total, the 120-cell has 600 vertices, 1200 edges and 720 pentagonal faces. Take a hyperbolic copy $P \subset H^4$ of the 120-cell in which the dihedral angles are $2\pi/5$. For each pair of opposite dodecahedral faces of P there is a unique reflection in a hyperplane which identifies them. Gluing opposite faces via these reflections gives the Davis manifold, a hyperbolic 4-manifold M.

The central involution of H^4 which fixes the centre of P preserves both P and the identifications of opposite faces, hence it gives an isometric involution σ of M. Our symplectic construction will begin with the resulting orbifold M/σ, which we call the *Davis orbifold*.

To analyse the fixed points of σ it is helpful to use the so-called "inside-out" isometry of M (defined in [10]). To describe this, note that P can be divided up into 14400 hyperbolic Coxeter simplexes. The vertices of a simplex are given by taking first the centre of P, then the centre of one of its 120 3-faces F, then the centre of one of the 12 2-faces f of F, then the centre of one of the 5 edges e of f and, finally, one of the two vertices of e. Denote by v_1, v_2, v_3, v_4, v_5 one such choice. The corresponding simplex has a isometry that exchanges v_1 (the centre of P) with v_5 (a vertex of e), v_2 (the centre of F) with v_4 (the centre of an edge of F') and fixes v_3 (the centre of f). This isometry of the simplex extends to define the inside-out isometry of M, which commutes with σ.

Lemma 6. *The fixed set of σ consists of 122 points. The quotient M/σ is simply connected as a topological space.*

Proof. In the interior of the 120-cell there is only one fixed point, the centre, all other fixed points of σ lie on the image in M of the boundary of P. Let F denote the image in M of a three-dimensional face of P; σ preserves F and induces on it the symmetry of the dodecahedron given by inversion $x \mapsto -x$ with respect to its centre. So, once again, in the interior of F there is only one fixed point, its centre. Considering all opposite pairs of three-dimensional faces of M this gives 60 more fixed points of σ. All remaining fixed points are contained in the image in M of the union of the 2-faces of P.

The symmetry σ takes 2-faces to 2-faces. We claim next that σ does not fix an interior point of any pentagonal 2-face. Assume for a moment that it does fix such a point. Then it would give an involution of the pentagon which would hence fix a vertex and so also the line joining the vertex to the centre of the polygon. The Davis manifold has two distinguished points, the centre and the image of all the vertices of the 120-cell. The assumption that σ fixes an interior point of a pentagonal 2-face gives a σ-fixed tangent direction at the vertex point in M. However, the inside-out isometry exchanges the centre and vertex of M. Since σ acts as $x \mapsto -x$ at the centre it does so also at the vertex and hence acts freely on the tangent space there. It follows that σ does not fix an interior point of any 2-face.

The remaining fixed points are contained in the image in M of the union of the edges of P. Under the inside-out involution of M, the middles of all edges are exchanged with centres of all 3-faces whilst the centre of P is exchanged with the image in M of the vertices of P. Since the inside-out isometry commutes with σ, this give an additional 61 fixed points of σ making 122 in total.

We now turn to the (topological, not orbifold) fundamental group of M/σ. The map $\pi_1(M) \to \pi_1(M/\sigma)$ is surjective so we need to show its image is trivial. Consider the 60 closed geodesics γ_i in M going through the centre of P and joining the centres of opposite faces. The deck transformations corresponding to these geodesics generate the whole of $\pi_1(M)$. Indeed, these deck-transformations take the fundamental domain P to all its 120 neighbours. Now the result follows from the fact that every loop $\sigma(\gamma_i)$ is contractible. □

3.3 The model singularity and resolution

Locally, the singularities of M/σ are modelled on the quotient of H^4 by $x \mapsto -x$. (Here x is the coordinate provided by the Poincaré ball model of H^4.) This lifts to action on $Z \to H^4$ where the fixed locus is an S^2-fibre.

To understand the resulting singularity in the quotient of Z we will use the Kähler description of Proposition 5. This says that we can identify Z symplectically with $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In this picture, the involution acts by $z \mapsto -z$ in the vector-bundle fibres of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$. The quotient $\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2$ is a Kähler Calabi–Yau orbifold with singular locus $\mathbb{CP}^1$ corresponding to the zero section. We next describe a crepant resolution of this singularity.

Lemma 7. *There is a crepant resolution*

$$\mathcal{O}(-2,-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2$$

where $\mathcal{O}(-2,-2) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ is the tensor product of the two line bundles given by pulling back $\mathcal{O}(-2) \to \mathbb{CP}^1$ from either factor.

Proof. Blow up the zero section of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ to obtain the total space of $\mathcal{O}(-1,-1) \to \mathbb{CP}^1 \times \mathbb{CP}^1$. The $\mathbb{Z}_2$-action lifts to this line bundle where it again has fixed locus the zero section and acts by $z \mapsto -z$ in the fibres. For such an involution on *any* line bundle L, the square gives a resolution $L^2 \to L/\mathbb{Z}_2$. Hence

$$\mathcal{O}(-2,-2) \to \mathcal{O}(-1,-1)/\mathbb{Z}_2 \to \mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2$$

gives the claimed resolution $\square$

We should emphasise here that this is resolution is *holomorphic*. The total space of $\mathcal{O}(-2,-2)$ is a Kähler manifold with trivial canonical bundle and, away from the exceptional divisor, the map in Lemma 7 is a biholomorphism when we consider $\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2$ with its *holomorphic* complex structure.

However, when constructing symplectic six-manifolds from hyperbolic four-manifolds, the relevant almost complex structure and volume form on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (and its $\mathbb{Z}_2$-quotient) are not the holomorphic ones; rather we use the $\mathrm{SO}(4,1)$-invariant almost complex structure and complex volume form from §2.1 and Remark 4. Lemma 7 can only be used to provide crepant resolutions of Calabi–Yau singularities modelled on the *holomorphic* geometry of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2$ and *not* the $\mathrm{SO}(4,1)$-invariant almost complex structure and complex volume form of §2.1.

So, in order to apply Lemma 7 to resolve the singularities in a hyperbolic twistor space, we need to interpolate between the holomorphic structures near the zero section in $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ to the $\mathrm{SO}(4,1)$-invariant structures outside a small neighbourhood of the zero section. This interpolation is provided by the following result.

Lemma 8. *Let Z_δ denote the part of $Z \to H^4$ lying over a geodesic ball in H^4 of radius δ. For any $\delta > 0$, there is an $\mathrm{SO}(4)$-invariant compatible almost complex structure J on Z and an $\mathrm{SO}(4)$-invariant nowhere-vanishing section Ω of the J-canonical-bundle such that:*

- *Over Z_δ, J and Ω agree with the standard holomorphic structures.*
- *Over $Z \setminus Z_{2\delta}$, J and Ω agree with the $\mathrm{SO}(4,1)$-invariant structures from §2.1 and Remark 4.*

Proof. As is standard, an $\mathrm{SO}(4)$-invariant interpolation between the "inside" and "outside" Hermitian metrics gives the existence of J.

To produce Ω we start with a description of the $\mathrm{SO}(4)$-action away from the zero-section Z_0. The stabiliser of a point $p \in Z \setminus Z_0$ is a circle $S^1_p \subset \mathrm{SO}(4)$ and the orbit of p is 5-dimensional (in fact, isomorphic as an $\mathrm{SO}(4)$-space to the unit tangent bundle of S^3). The lift of a geodesic ray out of the origin in H^4 meets each $\mathrm{SO}(4)$-orbit in a unique point, giving a section for the action. We interpolate between the holomorphic and hyperbolic complex volume forms along the relevant portion of this lifted ray and then use the $\mathrm{SO}(4)$-action to extend the resulting 3-form to the whole of Z. In order for this to work it is sufficient that at every point $p \in Z \setminus Z_0$ the action of S^1_p on the fibre of the J-canonical-bundle at p is trivial. But since the weight is integer valued and continuous it is constant on $Z \setminus Z_0$ so we can compute it for some p outside of Z_{2r} where everything agrees with the hyperbolic picture. Here we already have an SO(4)-invariant (hence S^1_p-invariant) complex volume-form so the weight is zero as required. $\square$

3.4 The twistor space of the Davis orbifold

With Lemmas 7 and 8 in hand, we can now take a crepant resolution of the twistor space of the Davis orbifold M/σ. Let $X \to M$ denote the twistor space of the Davis manifold The involution σ lifts to an involution of X which we still denote σ. X/σ is a symplectic orbifold with singularities along 122 $\mathbb{CP}^1$s, each modelled on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2$.

Let δ be a positive number small enough that the geodesic balls in M of radius 2δ centred on the σ-fixed points are embedded and disjoint. Then, by Lemma 8, on X we can find a new almost complex structure J and complex volume form Ω such that outside the geodesic 2δ-balls they agree with the hyperbolic structures coming from §2.1 and Remark 4, whilst inside the balls of radius δ they agree with the holomorphic structures coming from the holomorphic geometry of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. It follows from the SO(4)-invariance in Lemma 8 that J and Ω are σ-invariant.

In this way the quotient X/σ is a symplectic orbifold with an almost complex structure and complex volume form which are modelled near the singular curves on the holomorphic geometry of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2$. It follows from Lemma 7 that there is a resolution $\hat{X} \to X/\sigma$ in which the singular curves have been replaced by copies of $\mathbb{CP}^1 \times \mathbb{CP}^1$ with normal bundle $\mathcal{O}(-2, -2)$; moreover, $\hat{X}$ carries an almost complex structure $\hat{J}$ and complex volume form $\hat{\Omega}$ so that $c_1(\hat{X}, \hat{J}) = 0$.

Finally we need to define the symplectic structure on $\hat{X}$. Pulling back the symplectic form via $\hat{X} \to X$ gives a symplectic form on the complement of the exceptional divisors. To extend it we use a standard fact about resolutions in Kähler geometry. Given any neighbourhood U of the zero locus in $\mathcal{O}(-2, -2)$, there is a Kähler metric on $\mathcal{O}(-2, -2)$ for which the projection to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2$ is an isometry on the complement of U. (This amounts to the fact that the zero locus has negative normal bundle.)

So, in the model, the pull-back of the symplectic form extends over the exceptional divisor in a way compatible with holomorphic complex structure. Taking U sufficiently small and doing this near all 122 exceptional divisors defines a symplectic form ω on $\hat{X}$ which is compatible with $\hat{J}$.

3.5 Non-Kählerity

It remains to prove that there is no compatible Kähler structure on $\hat{X}$; indeed there is no Kähler structure whatsoever with $c_1 = 0$. This will follow from the fact that $\pi_1(\hat{X})$ and $b_3(\hat{X})$ both vanish. To prove simple-connectivity we first recall a standard lemma.

Lemma 9. *Let A and B be two finite dimensional CW complexes and let $f\colon A \to B$ be a surjective map with connected fibres. Suppose that B has an open cover by sets U_i such that for any $y \in U_i$ the inclusion homomorphism $\pi_1(f^{-1}(y)) \to \pi_1(f^{-1}(U_i))$ is an isomorphism. Then the following sequence is right-exact:*

$$\pi_1(f^{-1}(y)) \to \pi_1(A) \to \pi_1(B) \to 0.$$

(This is a truncated version of the long exact sequence of homotopy groups associated to a Serre fibration. The proof is identical.)

Lemma 10. *$\hat{X}$ is simply connected.*

Proof. We first apply Lemma 9 to the map $X/\sigma \to M/\sigma$. The fibres are S^2s and we see that $\pi_1(X/\sigma) = 1$. Next we apply Lemma 9 to $\hat{X} \to X/\sigma$. This time the fibres are points or S^2s and we deduce that $\pi_1(\hat{X}) = 1$. □

To prove that $b_3(\hat{X}) = 0$ we invoke a lemma of McDuff on the cohomology of manifolds obtained by symplectic blow-ups.

Lemma 11 (McDuff [9])**.** *Let X be a symplectic manifold and $C \subset X$ a smooth symplectic submanifold of codimension $2k$. Let $\tilde{X}$ denote the blow-up of X along C. Then the real cohomology of $\tilde{X}$ fits into a short exact sequence of graded vector spaces*

$$0 \to H^*(X) \to H^*(\tilde{X}) \to A^* \to 0$$

where the first arrow is pull-back via $\tilde{X} \to X$ and where A^* is free module over $H^*(C)$ with one generator in each dimension $2j$, $1 \le j \le k$.

Lemma 12. *$b_3(\hat{X}) = 0$.*

Proof. Recall that $X \to M$ is the twistor space of the Davis manifold. We first blow up the 122 fibres which lie over the fixed points of σ to obtain the new manifold $\tilde{X}$. It follows from Lemma 11 that pulling back cohomology via $\tilde{X} \to X$ induces an isomorphism $H^3(\tilde{X}) \cong H^3(X)$.

Next, notice that σ lifts to $\tilde{X}$ and that $\hat{X} = \tilde{X}/\sigma$. We now show that σ acts as -1 on $H^3(\tilde{X})$. To see this, consider the action of σ on the Davis manifold M. It acts on $H^1(M)$ as -1 and hence also as -1 on $H^3(M)$. Now $X \to M$ is a sphere-bundle so, by Leray–Hirsch, $H^*(X)$ is a free module over $H^*(M)$ with a single generator in degree 2 corresponding to the first Chern class of the vertical tangent bundle. This generator is preserved by

σ, so σ acts as -1 on $H^3(X)$ and hence also as -1 on $H^3(\tilde{X})$. From this we deduce that $H^3(\hat{X}) = 0$. For if it contained a non-zero element, the pull-back to $\tilde{X}$ would be a σ-invariant element of $H^3(\tilde{X})$. $\square$

Lemma 13. *A Kähler manifold with $b_1 = 0$ and $c_1 = 0$ has $b_2 \geq 2$.*

Proof. For a Kähler manifold, the vanishing of b_1 implies the Picard torus is trivial. I.e., a holomorphic line bundle which is topologically trivial bundle is necessarily holomorphically trivial. Now this combined with $c_1 = 0$ implies the canonical bundle is holomorphically trivial and so there is a holomorphic volume form. The real and imaginary parts of this form have independent cohomology classes and hence $b_3 \geq 2$. $\square$

Corollary 14. *$\hat{X}$ admits no Kähler structure for which $c_1 = 0$.*

4 Generalising the construction

The last section of [7] describes a variety of questions which arise out of this construction. We restrict ourselves here to the following one: is it possible to use the same approach—hyperbolic orbifolds and resolution of singularities—to produce more examples of symplectic manifolds with vanishing first Chern class?

A collection of hyperbolic orbifolds is mentioned in [7] with relatively simple singularities (although not as simple as the isolated singularities of the Davis orbifold). In forthcoming work we will show how the corresponding symplectic orbifolds can be resolved [6]. This produces a collection of simply-connected symplectic manifolds with $c_1 = 0$ and with arbitrarily large Betti numbers. It is as yet unknown whether a similar phenomenon can occur for Kähler Calabi–Yau manifolds.

If one was optimistic, however, one might hope that the potential for this construction is far greater. We need two things: the hyperbolic orbifolds and a way to resolve the resulting symplectic singularities. On the first topic, we mention a question due to Gromov: *Is every compact n-manifold homeomorphic to the quotient of H^n by a discrete group of isometries?* (Of course, the group is allowed to have torsion.) In dimensions 2 and 3 this is already known to be true. In dimension 4 this problem is seemingly wide open.

On the second topic—crepant resolutions—we remark that for algebraic threefolds, crepant resolutions are always known to exist by a famous result of Bridgeland, King and Reid [2]. Even independently of the intended application described here, it would be interesting to know what holds in the symplectic setting.

If one were ambitious, one might hope to use this sort of construction to approach the question of what groups might appear as the fundamental group of a symplectic 6-manifold with $c_1 = 0$. Can any finitely presented group arise this way?

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