Chapter 1

A first look at Kähler manifolds

In this section we introduce complex and Kähler manifolds. We give examples and introduce some of the basic tools used in their study, namely (p, q)-forms and Dolbeault cohomology. We finish with the Hodge theorem for Kähler manifolds, which relates Dolbeault cohomology and de Rham cohomology.

1.1 Definition and examples of complex manifolds

Definition 1.1. Let X be a manifold of dimension $2n$. An atlas $\{(\phi_\alpha, U_\alpha)\}$ for X is called *holomorphic* if when we think of the charts as taking values in $\mathbb{C}^n$ the transition functions $\phi_{\alpha\beta} \colon \phi_\alpha(U_\alpha) \to \phi_\beta(U_\beta)$ are biholomorphisms (i.e., holomorphic homeomorphisms with holomorphic inverses). In this case we call X a complex manifold.

Note that we can now make sense of holomorphic functions $X \to \mathbb{C}$ from a complex manifold (by checking in a holomorphic atlas, just as in the definition of smooth functions from a real manifold). We can also talk about holomorphic maps of complex manifolds.

Examples 1.2.

1. $\mathbb{C}^n$ or open subsets thereof are complex manifolds covered by a single chart. For example, $\mathrm{GL}(k, \mathbb{C}) \subset \mathbb{C}^{k^2}$ is the open subset of $k \times k$ matrices where the determinant is non-zero and hence is a complex manifold.

2. One can find complex submanifolds using the implicit function theorem just as for smooth manifolds. For example, $\det \colon \mathrm{GL}(k, \mathbb{C}) \to \mathbb{C}$ is a holomorphic function with surjective derivative on the subset $\mathrm{SL}(k, \mathbb{C})$ of matrices with determinant 1. It follows that $\mathrm{SL}(k, \mathbb{C})$ is itself a complex manifold.

3. Generalising the previous example are *affine varieties*. An affine variety is a set

$X \subset \mathbb{C}^n$ which is locally the common zero-locus of a finite collection of holomorphic functions. I.e., there is an open cover of $\mathbb{C}^k$ such that for each U in the cover there are holomorphic functions $f_1, \dots, f_k$ such that $X \cap U = \{z : f_j(z) = 0,\ j = 1, \dots, k\}$.

If in addition one can always arrange that the number k of these functions is independent of U and moreover that the differentials $\mathrm{d}f_j$ give a surjection $\mathbb{C}^N \to \mathbb{C}^k$ at each point of $X \cap U$ then the implicit function theorem implies that X is a complex manifold.

For example, the subset $\{z : \sum z_j^2 = \epsilon\}$ of $\mathbb{C}^n$ is an affine variety, which is smooth for $\epsilon \neq 0$.

4. $\mathbb{CP}^n$ is a complex manifold. Write $U_j = \{[z_0 : \cdots : z_n] : z_j \neq 0\}$ and define $\phi_j \colon U_j \to \mathbb{C}^n$ by $\phi_j[z_0 : \cdots : z_n] = (z_0/z_j, \dots, z_{j-1}/z_j, z_{j+1}/z_j, \dots, z_n)$. One can check directly that the transition functions are holomorphic, making $\mathbb{CP}^n$ a complex manifold.

5. Just as we defined affine varieties as submanifolds of $\mathbb{C}^n$, so we can define projective varieties as submanifolds of $\mathbb{CP}^n$.

   When considering $\mathbb{C}^n$, we could look at the zero locus of a single holomorphic function, but in $\mathbb{CP}^n$ this is no longer possible, since $\mathbb{CP}^n$ is compact, the only global holomorphic functions are constants. (See the exercises for the chance to prove this yourself.)

   Instead, one can use *homogeneous polynomials* in place of holomorphic functions. If p is a homogeneous polynomial in $n+1$-variables, with complex coefficients, then whilst p does not make sense as a function on $\mathbb{CP}^n$, its zero locus does. This is because if $p(z_0, \dots, z_n) = 0$ then so does $p(\lambda z_0, \dots, \lambda z_n) = \lambda^d p(z_0, \dots, z_n)$ (where d is the degree of p).

   So to each homogeneous polynomial p, we can associate $X_p = \{[z_0, \dots, z_n] : p(z_0, \dots, z_n) = 0\}$, the hypersurface cut out by p. Again one can use the implicit function theorem to determine when X_p is in fact smooth.

   More generally, one can look at subsets of $\mathbb{CP}^n$ which are locally the common zero locus of a collection of homogeneous polynomials.

6. Let (X, g) be an oriented surface (real dim 2) with a Riemannian metric. Isothermal coordinates for X are coordinates (x, y) in which the metric has the form $g = f(x, y)(\mathrm{d}x^2 + \mathrm{d}y^2)$ for some function f, i.e., g is conformally equivalent to the Euclidean metric. It is an non-trivial fact that such coordinates always exist. Now a diffeomorphism $\mathbb{C} \to \mathbb{C}$ is holomorphic if and only if it is conformal. This means that the transition maps between isothermal coordinate charts are exactly those which are holomorphic with respect to the variable $z = x + iy$ and so an atlas of isothermal charts is a holomorphic atlas on X.

(Note this equivalence of holomorphic and conformal maps is particular to complex dimension 1. In higher dimensions the two are in some sense orthogonal concepts.)

7. One can take quotients of complex manifolds to obtain new ones. A simple example is the following: let $\tau \in \mathbb{C}$ have strictly positive imaginary part and consider $\Lambda_\tau = \{m + n\tau : m, n \in \mathbb{Z}\}$, an additive subgroup of $\mathbb{C}$ abstractly isomorphic to $\mathbb{Z}^2$. The quotient $X_\tau = \mathbb{C}/\Lambda_\tau$ is naturally a complex manifold in such a way that the covering map $\mathbb{C} \to X_\tau$ is holomorphic. These complex manifolds are called *elliptic curves*. ("Curves" since they have complex dimension 1.) As smooth manifolds they are all diffeomorphic (to T^2, the torus) but as complex manifolds they are in general different (see the exercises).

   As another example, consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus 0$ generated by $z \mapsto \lambda z$ where $\lambda \in \mathbb{C}^*$ is some fixed non-zero complex number. Again, the quotient $X_\lambda = (\mathbb{C}^2 \setminus 0)/\mathbb{Z}$ is a complex manifold, called a *Hopf surface*. ("Surface" because it has complex dimension 2.) They are all diffeomorphic to $S^3 \times S^1$, but as complex manifolds they are in general different.

Just as one can talk of smooth functions, smooth vector bundles, smooth sections etc. on a differentiable manifold, one can also talk of holomorphic functions, holomorphic vector bundles, holomorphic sections etc. on a complex manifold. E.g., with respect to a local trivialisation, the transition functions of a holomorphic vector bundle are *holomorphic* maps $U_{\alpha\beta} \to \mathrm{GL}(k, \mathbb{C})$.

The most obvious example of a holomorphic vector bundle on a complex manifold is its tangent bundle. The fist step is make TX into a complex vector bundle. To explain this we begin with a short digression. A map $f \colon \mathbb{C}^n \to \mathbb{C}^k$ is holomorphic precisely when its derivative $\mathrm{d}f_x \colon \mathbb{C}^n \to \mathbb{C}^k$ is complex linear for each x. (For some people this is the definition of holomorphic; what ever definition you use, it should be easy to show it's equivalent to this). This means that when X is a complex manifold one can define an endomorphism $J \colon T_xX \to T_xX$ with $J^2 = -1$ as follows. Pick any chart which contains x; in this chart, $T_xX \cong \mathbb{C}^n$ and define J to be $\times i$ under this identification; changing the chart changes the identification $T_xX \cong \mathbb{C}^n$ by the derivative of a transition function; since this is complex linear it doesn't change the meaning of "$\times i$" and so J is independent of the choice of chart. In other words, the tangent spaces of a complex manifold are naturally complex vector spaces; multiplication by i on each tangent space giving the endomorphism J. Now that TX is a complex vector bundle it makes sense to ask if it is holomorphic. It is an exercise to check that the transition functions for TX inherited from those of an atlas are indeed holomorphic maps from their domains to $\mathrm{GL}(n, \mathbb{C})$.

As an aside, given a complex manifold X one can recover a holomorphic atlas from J: the complex structure is all you need to be able to define the holomorphic functions, given $U \subset X$ open, a map $f\colon U \to \mathbb{C}$ is holomorphic iff $\mathrm{d}f \circ J = i\mathrm{d}f$. For this reason, we will often write a complex manifold as (X, J).

Exercises 1.3.

1. Check that the tangent bundle of a complex manifold is a holomorphic vector bundle.

2. Prove that on a compact complex manifold, the only holomorphic functions $f\colon X \to \mathbb{C}$ are the constant functions.

   Hint: Note that, away from a zero of f, one can define locally the logarithm, $\log f = \log|f| + i \arg f$. *Now apply the maximum principle to the real part* $\log|f|$ *of this holomoprhic function.*

3. Under what circumstances are the elliptic curves X_τ and $X_{\tau'}$ (from Examples 1.2.7) biholomorphic? (Hint: given a biholomorphism $X_\tau \to X_{\tau'}$ consider the lift to the universal covers $\mathbb{C} \to \mathbb{C}$ and show that it must send Λ_τ to $\Lambda_{\tau'}$.)

1.2 Differential forms on complex manifolds

The complex structure J induces a decomposition of differential forms which is important in the study of complex manifolds. We look first at a complex-valued 1-form $\alpha \in C^\infty(T^*X \otimes \mathbb{C})$, where we take the tensor product over $\mathbb{R}$. That is to say that at a point x, α_x is a *real*-linear map $\alpha_x\colon T_xX \to \mathbb{C}$. The motivation for this is that whilst differentials of real valued functions are sections of T^*X, the differential of a complex valued function is a section of $T^*X \otimes \mathbb{C}$.

Since both T_xX and $\mathbb{C}$ are complex vector spaces we can decompose $\alpha \in T^*_xX \otimes \mathbb{C}$ into complex-linear and complex-anti-linear parts, denoted $\alpha^{1,0}$ and $\alpha^{0,1}$ respectively: $\alpha = \alpha^{1,0} + \alpha^{0,1}$ where

$$\alpha^{1,0}(u) = \frac{1}{2}\left(\alpha(u) - i\alpha(Ju)\right), \qquad \alpha^{0,1}(u) = \frac{1}{2}\left(\alpha(u) + i\alpha(Ju)\right).$$

This decomposition gives a splitting of the complexified cotangent bundle into complex-linear and complex-anti-linear pieces:

$$T^*X \otimes \mathbb{C} = T^*X^{1,0} \oplus T^*X^{0,1}.$$

$T^*X \otimes \mathbb{C}$ carries a conjugation map: $\alpha \otimes z \mapsto \alpha \otimes \bar{z}$, and that this exchanges the summands in the above splitting. The fixed points of the conjugation map are those vectors of the form $\alpha \otimes 1$, giving an embedding $T^*X \to T^*X \otimes \mathbb{C}$ of the *real* cotangent bundle into the *complex* cotangent bundle. One can check from the above formula for $\alpha^{1,0}$ that $\alpha \mapsto (\alpha \otimes 1)^{1,0}$ is a complex-linear isomorphism $(T^*X, J) \to T^*X^{1,0}$.

Definition 1.4. $T^*X^{1,0}$ is called the *holomorphic cotangent bundle*, which is genuinely a holomorphic bundle via the above identification with (T^*X, J). $T^*X^{0,1}$ is called the *anti-holomorphic cotangent bundle*. A $(1,0)$*-form* is a smooth section of the bundle $T^*X^{1,0}$; similarly a $(0,1)$*-form* is a smooth section of the bundle $T^*X^{0,1}$.

Example 1.5. As a simple example, consider $\mathbb{R}^2 \cong \mathbb{C}$ as a complex manifold. We have tangent vectors ∂_x, ∂_y corresponding to the real and imaginary axes in $\mathbb{C}$ and $J(\partial_x) = \partial_y$, $J(\partial_y) = -\partial_x$. Dually, $J(\mathrm{d}x) = -\mathrm{d}y$ and $J(\mathrm{d}y) = \mathrm{d}x$. Both $T^*X^{1,0}$ and $T^*X^{0,1}$ are rank 1. Trivialisations are given by

$$2(\mathrm{d}x)^{1,0} = \mathrm{d}x - iJ\mathrm{d}x = \mathrm{d}x + i\mathrm{d}y, \quad 2(\mathrm{d}x)^{1,0} = \mathrm{d}x + iJ\mathrm{d}x = \mathrm{d}x - i\mathrm{d}y.$$

Writing $z = x + iy$ we have $\mathrm{d}z = 2(\mathrm{d}x)^{1,0}$ and $\mathrm{d}\bar{z} = 2(\mathrm{d}x)^{0,1}$.

More generally, on $\mathbb{C}^n$ with coordinates $(z_1, \dots, z_n)$, the $(1,0)$-forms $\mathrm{d}z_1, \dots, \mathrm{d}z_n$ span the holomorphic cotangent bundle whilst the $(0,1)$-forms $\mathrm{d}\bar{z}_1, \dots, \mathrm{d}\bar{z}_n$ span the anti-holomorphic cotangent bundle.

The splitting of the complex cotangent bundle induces in turn a splitting of the exterior derivative. Define $\partial\colon C^\infty(X, \mathbb{C}) \to C^\infty(T^*X^{1,0})$ by $\partial f = (\mathrm{d}f)^{1,0}$ and $\bar{\partial}\colon C^\infty(X, \mathbb{C}) \to C^\infty(T^*X^{0,1})$ by $\bar{\partial} f = (\bar{\partial} f)^{0,1}$.

In local coordinates $z_j = x_j + iy_j$,

$$\partial f = \sum \frac{\partial f}{\partial z_j} \mathrm{d}z_j, \qquad \bar{\partial} f = \sum \frac{\partial f}{\partial \bar{z}_j} \mathrm{d}\bar{z}_j$$

where

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right), \qquad \frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right)$$

Note that a function is holomorphic if and only if $\bar{\partial} f = 0$.

The splitting of the cotangent bundle induces in turn a splitting of all exterior powers. Write $\Lambda^{p,q} = \Lambda^p(T^*X^{1,0}) \otimes \Lambda^q(T^*X^{0,1})$. Then

$$\Lambda^r T^*X \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}$$

In coordinates, $z_j = x_j + iy_j$, there is a basis $\{\mathrm{d}z_j, \mathrm{d}\bar{z}_j\}$ for $T^*X \otimes \mathbb{C}$. Then

$$\{\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J : |I| + |J| = r\}$$

is a basis for $\Lambda^r T^*X \otimes \mathbb{C}$ whilst $\Lambda^{p,q}$ has as a basis

$$\{\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J : |I| = p,\ |J| = q\}.$$

Here $I = (i_1, \dots, i_p)$ is a multi-index with $1 \leq i_1 < \cdots < i_p$ and $\mathrm{d}z_I = \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p}$.

Definition 1.6. A (p,q)-form is a smooth section of the bundle $\Lambda^{p,q}$. The space of all (p,q)-forms is denoted $\Omega^{p,q}$ or $\Omega^{p,q}(X)$ when it is necessary to specify the underlying complex manifold.

Just as the exterior derivative on functions extends to differential forms, so ∂ and $\bar{\partial}$ extend to (p,q)-forms. By definition, $\bar{\partial}\colon \Omega^{p,q} \to \Omega^{p,q+1}$ is the $(p,q+1)$-projection of d and $\partial\colon \Omega^{p,q} \to \Omega^{p+1,q}$ is the $(p+1,q)$-projection of d. One can check (for example in local coordinates) that there are no other components of $d\colon \Omega^{p,q} \to \Omega^{p+q+1}$.

We saw above that functions f with $\bar{\partial} f$ are simply the holomorphic functions. There is an analogous interpretation for higher degree $(p,0)$-forms

Definition 1.7. $\Lambda^{p,0} = \Lambda^p(T^*X^{1,0})$ is a holomorphic vector bundle, called *the bundle of holomorphic p-forms*. A section $s \in \Omega^{p,0}$ is holomorphic precisely when $\bar{\partial} s = 0$. In this case we say that s is a holomorphic p-form.

Just as for the exterior derivative, $\bar{\partial}^2 = 0$ and so one can define Dolbeault cohomology, by analogy with de Rham cohomology.

Definition 1.8. The (p,q)-Dolbeault cohomology group of X is the vector space

$$H^{p,q}(X) = \frac{\ker \bar{\partial}\colon \Omega^{p,q} \to \Omega^{p,q+1}}{\operatorname{Im} \bar{\partial}\colon \Omega^{p,q-1} \to \Omega^{p,q}}$$

Notice that $H^{p,0}$ is simply the space of holomorphic p-forms.

Exercises 1.9.

1. (a) Prove that $\bar{\partial}^2 = 0$.
   (b) Prove that $\bar{\partial}\partial + \partial\bar{\partial} = 0$.
   (c) Check that $\mathrm{d}\colon \Omega^{p,q} \to \Omega^{p+1,q} \oplus \Omega^{p,q+1}$.
2. Let X be a simply connected compact complex manifold. Prove that $H^{1,0}(X) = 0$.
   Hint: given a holomorphic 1-form α, integrate it along paths with a fixed start-point to define a holomorphic map $f\colon X \to \mathbb{C}$ with $\mathrm{d}f = \alpha$.
3. (a) Find an example of a compact complex manifold with $H^{1,0}(X)$ non-zero.
   (b) For each $n \in \mathbb{N}$, find an example of a compact complex manifold of complex dimension n for which $H^{p,0}(X)$ is non-zero for $p = 0, \dots, n$.

1.3 Definitions and examples of Kähler manifolds

Put briefly (and somewhat pompously!) Kähler geometry is the harmonious combination of complex and Riemannian geometry. In more detail, let (X, J) be a complex manifold. The first way in which a Riemannian metric g on X can be compatible with J is a pointwise algebraic condition.

Definition 1.10. A Riemannian metric g on X is called *Hermitian* if $g(Ju, Jv) = g(u, v)$ for all $u, v \in TX$.

This is equivalent to saying that the bilinear form $\omega(u, v) = g(Ju, v)$ is skew and of type $(1, 1)$. The fact that g is positive definite implies that ω is positive on all complex lines.

Definition 1.11. A real $(1, 1)$-form is called *positive* if it is positive on all complex lines, i.e., $\omega(u, Ju) > 0$ for all $u \neq 0$.

Notice that g can be recovered from ω and J via $g(u, v) = \omega(u, Jv)$. This means that specifying a Hermitian metric g on X is the same thing as specifying a positive $(1, 1)$-form ω.

Definition 1.12. Given a Hermitian metric g, we call ω the *associated* $(1, 1)$*-form* of g.

A Kähler manifold is a complex manifold with a Hermitian metric which also satisfies a *differential* compatibility condition.

Proposition 1.13. *Let* (X, J, g) *be a Hermitian manifold. The following are equivalent:*

1. *The complex structure* J *is parallel with respect to the Levi-Civita connection:* $\nabla J = 0$.
2. *The associated* $(1, 1)$*-form* ω *is parallel:* $\nabla\omega = 0$.
3. *The associated* $(1, 1)$*-form* ω *is closed:* $\mathrm{d}\omega = 0$.
4. *Locally, one can write* $\omega = i\bar{\partial}\partial\phi$ *for a real valued function* ϕ*, called a* local Kähler potential.
5. *There exist holomorphic coordinates* $z_1, \ldots, z_n$ *in which the metric is Euclidean to second order:* $g = \sum \mathrm{d}z_i \otimes \mathrm{d}\bar{z}_i + O(|z|^2)$.

The proof of this is an exercise.

Definition 1.14. When one, and hence all, of the above conditions are met we call (X, J, g) a *Kähler* manifold. (We will equally write a Kähler manifold as (X, J, ω) when we have the positive $(1, 1)$-form in mind, or even (X, ω) when the underlying complex structure is implicit.)

Note that any complex manifold admits a Hermitian metric (the proof is identical to that of the existence of Riemannian metrics). The existence of a Kähler metric however is a far more subtle question. We will see later some important topological obstructions; there are others (notably concerning the fundamental group) which we do not have time to discuss. But these conditions aside, determining whether a given complex manifold is Kähler or not is an extremely hard problem (often impossible with current techniques).

Examples 1.15.

1. Let (X, J) be a Riemann surface (i.e., complex manifold of dimension 1) and let g be a Hermitian metric with associated $(1, 1)$-form ω. Since there are no 3-forms on a surface, $\mathrm{d}\omega = 0$ and (X, J, g) is Kähler.

2. Let (X, g) be an oriented surface (real dim 2) with a Riemannian metric. We saw above that isothermal coordinate charts for g define a holomorphic atlas which makes X into a complex manifold. One can check that $J \colon TX \to TX$ is simply a positive rotation by $\pi/2$. It follows that g is Hermitian with respect to J and so (X, J, g) is Kähler as above.

3. Let (X, J, ω) be Kähler and $Y \subset X$ a complex submanifold. The restriction of the Kähler metric to Y has associated $(1, 1)$-form given by the restriction of ω. Since ω is closed, so too is its restriction. Hence the induced metric on Y is again Kähler.

4. Fix a Hermitian innerproduct on $\mathbb{C}^{n+1}$. Then $\mathbb{CP}^n$ inherits a canonical (up to scale) $\mathrm{U}(n + 1)$-invariant Riemannian metric, called the Fubini–Study metric. One way to see this is to regard $\mathbb{CP}^n$ as the quotient S^{2n+1}/S^1 of the unit sphere in $\mathbb{C}^{n+1}$. There is then a unique metric on $\mathbb{CP}^n$ which makes the quotient map $S^{2n+1} \to \mathbb{CP}^n$ a Riemannian submersion.

   The Fubini–Study metric is Kähler, as can be seen in various ways. One can either compute in a local unitary chart, to see that $\mathrm{d}\omega = 0$, or use symmetry arguments to see that $\nabla J = 0$. (See the exercises for one approach.)

5. The previous two observations combine to give a plethora of examples: *any complex submanifold of* $\mathbb{CP}^n$ *inherits a Kähler metric*. (Recall from above that there are many such submanifolds, given locally as the common zeros of homogeneous polynomials in $n + 1$ variables.)

An obvious topological invariant of a Kähler manifold (X, ω) is the cohomology class $[\omega] \in H^2(X, \mathbb{R})$ of the Kähler form, called *the Kähler class*.

Lemma 1.16. *On a Hermitian manifold (X, ω) of complex dimension n, the volume form is $\frac{\omega^n}{n!}$.*

Proof. Since this is a purely pointwise statemtent it suffices to check it for the flat metric on $\mathbb{C}^n$. □

Corollary 1.17. *If (X, ω) is a compact Kähler manifold then $[\omega]^n$ is non-zero in $H^{2n}(X, \mathbb{R})$.*

Corollary 1.18. *If two complex submanifolds of a Kähler manifold are homologous then they have the same volume.*

Exercises 1.19.

1. Prove the equivalence of the various definitions of Kähler by proving the chain of implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$ in Proposition 1.13.

   Hint: to prove $3 \Rightarrow 4$ use the Poincaré lemma which states that if α is a d-closed p-form then locally one can write $\alpha = \mathrm{d}\beta$ for a $(p-1)$-form, together with analogous results for ∂ and $\bar{\partial}$. To prove $4 \Rightarrow 5$ pick arbitrary coordinates at a point p, then rotate so that $i\bar{\partial}\partial\phi$ is diagonal at p; then scale so that it is the identity at p; then use a coordinate change fixing p and T_pX which eliminates the first order terms in $i\bar{\partial}\partial\phi$.

2. Consider the hyperbolic metric on the unit disc $D = \{|z| < 1\}$ given by

   $$g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{(1 - x^2 - y^2)^2}$$

   Find a global function $\phi \colon D \to \mathbb{R}$ so that the associated $(1,1)$-form of g is given by $\omega = i\bar{\partial}\partial\phi$.

3. Let $U \subset \mathbb{CP}^n$ be an open set and $f \colon U \to \mathbb{C}^{n+1} \setminus 0$ a local section of the projection map. Prove that the $(1,1)$-form $\omega_{U,f} = -i\bar{\partial}\partial \log |f|$ is positive and that in fact it doesn't depend on the choice of section f.

   Deduce that there is a $\mathrm{U}(n+1)$-invariant Kähler metric on $\mathbb{CP}^n$ which agrees with each $\omega_{U,f}$.

   (This is another description of the Fubini–Study metric.)

4. Prove that there is a unique Riemannian metric on $\mathbb{CP}^n$, up to scale, which is invariant with respect to the action of $\mathrm{U}(n+1)$.

5. Prove that the Hopf surfaces of Example 1.2.7 do not admit Kähler metrics.

   Hint: what would the Kähler class be?

1.4 The Kähler identities and Hodge theory

Just as on a Riemannian manifold one can define the L^2-adjoint d^* of the exterior derivative in terms of the Hodge star $\mathrm{d}^* = \pm * \mathrm{d} *$, one can do similarly for ∂^* and $\bar{\partial}^*$ on a Hermitian manifold. Note that $\partial^* \colon \Omega^{p,q} \to \Omega^{p-1,q}$ whilst $\bar{\partial} \colon \Omega^{p,q} \to \Omega^{p,q-1}$.

One of the fundamental facts for Kähler manifolds is the interaction of these operators and the wedge-product with the Kähler form ω.

Definition 1.20. The map $L\colon \Lambda^p \to \Lambda^{p+2}$ defined by $L(\alpha) = \omega \wedge \alpha$ is called the *Lefschetz operator*.

Proposition 1.21 (The Kähler identities)**.** *On a Kähler manifold, the following hold*

$$[\bar{\partial}^*, L] = i\partial, \qquad [\partial^*, L] = -i\bar{\partial}.$$

To prove these identities, note first that they only see first order derivatives of the Kähler structure. This means that by part 6 of Proposition 1.13 that it suffices to prove them for the flat metric on $\mathbb{C}^n$.

On a Riemannian manifold, we can define the Laplacian on forms:

$$\Delta_{\mathrm{d}} = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$$

On a Hermitian manifold, we can do similarly with ∂ and $\bar{\partial}$:

$$\Delta_{\partial} = \partial^*\partial + \partial\partial^*, \qquad \Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$$

In general these Laplacians have little to do with each other, but on a Kähler manifold it is a corollary of the Kähler identities that they are all essentially one and the same:

Corollary 1.22. *On a Kähler manifold,*

$$\Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta_{\mathrm{d}}$$

Notice that, by definition, $\Delta_{\bar{\partial}}$ preserves bidegree of forms, i.e., it is a map $\Omega^{p,q} \to \Omega^{p,q}$. For an aribtrary Hermitian metric it is certainly not the case that the Riemannian Laplacian Δ_{d} preserves bidegree; the image $\Delta_{\mathrm{d}}(\Omega^{p,q})$ may well have many components of different bidegrees. But when the metric is also Kähler, the identity of Corollary 1.22 shows that Δ_{d} *does* preserve bidegree. This means that the pointwise decomposition of differential forms into (p,q)-forms passes to cohomology.

To see this we recall first the Hodge theorem for compact orientable Riemannian manifolds.

Theorem 1.23 (The Hodge theorem for Riemannian manifolds)**.** *Let (M, g) be a compact orientable Riemannian manifold. Each de Rham cohomolgy class $\kappa \in H^r(M, \mathbb{R})$ contains a unique harmonic representative, i.e., one satisfying $\Delta_{\mathrm{d}}\alpha = 0$. This gives an isomorphism $H^r(M, \mathbb{R}) \cong \mathcal{H}^r_{\mathrm{d}}(M, g)$ between degree r de Rham cohomolgy and the space of harmonic r-forms.*

For a Hermitian manifold, the same argument works for Dolbeault cohomolgy and the $\bar{\partial}$-Laplacian.

Theorem 1.24 (The Hodge theorem for Hermitian manifolds)**.** *Let (X, J, g) be a compact Hermitian manifold. Each Dolbeault cohomology class $\kappa \in H^{p,q}(X)$ contains a unique $\bar\partial$-harmonic representative, i.e., one satisfying $\Delta_{\bar\partial}\alpha = 0$. This gives an isomorphism $H^{p,q}(X) \cong \mathcal{H}^{p,q}_{\bar\partial}(X)$ between Dolbeault cohomology of bidegree (p,q) and the space of $\bar\partial$-harmonic (p,q)-forms.*

Thanks to Corollary 1.22, on a Kähler manifold the two notions of "harmonic" employed here agree. In the following result, $b_r = \dim H^r(X, \mathbb{R})$ is the r^{th} Betti number of X, whilst $h^{p,q} = \dim H^{p,q}(X)$ are the Hodge–Betti numbers of X.

Theorem 1.25 (The Hodge theorem for Kähler manifolds)**.** *On a Kähler manifold, a differential form is* d*-harmonic if and only if it is $\bar\partial$-harmonic. Hence*

$$\mathcal{H}^r_{\mathrm{d}}(X) = \bigoplus_{p+q=r} \mathcal{H}^{p,q}_{\bar\partial}(X)$$

It follows that $b_r = \sum_{p+q=r} h^{p,q}$.

Moreover, conjugation gives an isomorphism $\mathcal{H}^{p,q}_{\bar\partial}(X) \to \mathcal{H}^{q,p}_{\bar\partial}$ whilst the Hodge star gives an isomorphism $\mathcal{H}^{p,q}_{\bar\partial}(X) \to \mathcal{H}^{n-p,n-q}_{\bar\partial}(X)$. It follows that $h^{p,q} = h^{q,p} = h^{n-p,n-q}$.

Proof. Let α be d-harmonic. Since $\Delta_{\mathrm{d}} = 2\Delta_{\bar\partial}$ preserves bidegree, each (p,q)-component of α is itself d- and hence $\bar\partial$-harmonic. This means that the (p,q)-decomposition of forms passes to harmonic forms as claimed.

To prove the statement about conjugation, note that $\overline{\bar\partial\alpha} = \partial\bar\alpha$ and so $\overline{\Delta_{\bar\partial}\alpha} = \Delta_\partial\bar\alpha$. In particular if α is $\bar\partial$-harmonic then $\bar\alpha$ is ∂-harmonic. Since these notions coincide on a Kähler manifold the result follows.

Finally, to prove the statement about the Hodge star, it suffices to check that it maps $\Omega^{p,q}$ to $\Omega^{n-p,n-q}$ and that it commutes with the Laplacian(s). □

Corollary 1.26. *If X is a compact Kähler manifold, b_{2r+1} is even for all r.*

Proof. $b_{2r+1} = h^{2r+1,0} + h^{2r,1} + \cdots + h^{1,2r} + h^{0,2r+1} = 2(h^{2r+1,0} + h^{2r,1} + \cdots + h^{r+1,r})$. □

Exercises 1.27.

1. Prove the Kähler identities on $\mathbb{C}^n$ and hence on any Kähler manifold.
2. Prove the formulae in Corollary 1.22.
3. (a) Prove that on a Kähler manifold the following identity holds for all functions f.
$$\frac{1}{2}\Delta_{\mathrm{d}} f\, \omega^n = n\, i\bar\partial\partial f \wedge \omega^{n-1}$$
   (b) Prove that $\frac{1}{2}\Delta_{\mathrm{d}} f = \langle i\bar\partial\partial f, \omega\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the inner-product on forms.