Constant scalar curvature metrics on fibred complex surfaces

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Abstract

This thesis proves the existence of constant scalar curvature Kähler metrics on certain connected compact complex surfaces. The surfaces considered are those admitting a holomorphic submersion to curve, with fibres of genus at least 2. The proof is via an adiabatic limit. An approximate solution is constructed out of the hyperbolic metrics on the fibres and a large multiple of a certain metric on the base. A parameter dependent inverse function theorem is then used to perturb the approximate solution to a genuine solution in the same cohomology class.

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Contents

Introduction *1*

This thesis proves the existence of constant scalar curvature Kähler metrics on certain compact connected complex surfaces. The surfaces under consideration are those which admit a holomorphic submersion $\pi\colon X \to \Sigma$ onto a complex curve. Moreover, it is assumed that the fibres of π have high genus (*i.e.* greater than 1). That π is a surjective submersion says simply that, as a smooth manifold, X is a surface bundle over a surface. Since π is holomorphic, the fibres of π are all complex submanifolds. It is important to note that the complex structure of the fibres will, in general, vary. Examples of such surfaces will be given later (see section 1.2).

The first observation to make is that the hypotheses on X guarantee that it is projective. The dual of the vertical tangent bundle is a holomorphic line bundle whose fibrewise restriction is positive. It may not be positive transverse to the fibres. Adding a high power of a positive bundle pulled up from the base, however, gives a positive holomorphic line bundle, showing that X is projective. (An actual proof of the positivity of this line bundle is given in Lemma 4.1.) Denoting the vertical tangent bundle by V, the ample classes described here are of the form

$$\kappa_r = -2\pi \left(c_1(V) + rc_1(\Sigma)\right)$$

for large enough r. In the discussion of projectivity, it is necessary to take r to be a positive integer so that κ_r is an integral class. When r is a large positive real number, however, κ_r is still a Kähler class.

The precise result proved in this thesis is:

Theorem 1.1. *If X is a compact connected complex surface admitting a holomorphic submersion onto a complex curve with fibres of genus at least two, then, for all large r, the Kähler class κ_r contains a constant scalar curvature Kähler metric.*

The arguments should also apply more generally to higher dimensional fibred Kähler manifolds, although the conditions are more awkward to state. The higher dimensional version of Theorem 1.1 is stated in Chapter 9 (see Conjecture 9.1).

Section 1.1 motivates the search for constant scalar curvature metrics; Section 1.2 provides examples of surfaces to which Theorem 1.1 applies. Section 1.3 discusses the main analytic technique, the adiabatic limit, and explains other related results where this technique has been applied. Section

1.4 gives an overview of the proof of Theorem 1.1. Chapter 2 reviews results about constant scalar curvature metrics on high genus curves. In Chapter 3, general properties of scalar curvature are discussed. In Chapter 4, families of approximate solutions are constructed. Chapters 5, 6, 7 and 8 carry out the adiabatic limit which shows that these approximate solutions can be perturbed to genuine solutions. Finally, Chapter 9 discusses the difficulties in generalising the proof of Theorem 1.1 to include higher dimensional manifolds and manifolds admitting fibrations with singular fibres.

1.1 Constant scalar curvature metrics

In this section four related reasons are given for studying constant scalar curvature Kähler metrics.

1.1.1 Complex curves

The classical correspondence between constant scalar curvature metrics and complex structures plays a prominent role in the study of Riemann surfaces. This leads to, among other things, a geometric realisation of the complex moduli for curves. For example, in the high genus case, the moduli can be thought of as the lengths of $3g-3$ hyperbolic geodesics splitting the curve up into rigid pieces (with three cylindrical ends, homeomorphic to a sphere punctured in three places) and the size of the twists applied before putting the pieces back together ($6g-6$ real parameters in total).

This correspondence motivates the idea that constant scalar curvature Kähler metrics may be of use in algebraic geometry. The precise context in which such metrics should be important is described below in section 1.1.4.

The existence and uniqueness of constant scalar curvature metrics on compact high genus curves will be discussed in more detail in Chapter 2.

1.1.2 Extremal metrics

In [Cal82] and [Cal85] Calabi studies the functional which associates to each Kähler metric the L^2-norm of its scalar curvature. The Euler-Lagrange equations show that the critical points of this functional (when restricted to a single cohomology class) are metrics ω satisfying

$$\bar{\partial}\left(\nabla \operatorname{Scal}(\omega)\right) = 0,$$

i.e. those for which the gradient of the scalar curvature defines a holomorphic vector field. Such metrics are called *extremal*. In particular, constant scalar curvature Kähler metrics are extremal. It might be hoped that extremal metrics provide "canonical" representatives of a given Kähler class.

1.1.3 Kähler-Einstein metrics

A Kähler metric on a complex manifold X is said to be *Kähler-Einstein* if the Ricci form ρ and the Kähler form ω are proportional,

$$\rho = \lambda\omega$$

(where λ is a constant). There is an obvious topological condition which must be satisfied for such metrics to exist: the corresponding cohomology classes must be equal; *i.e.* the Kähler class $[\omega]$ must satisfy

$$\lambda[\omega] = 2\pi c_1(X).$$

If ω is Kähler-Einstein, it follows from taking traces that ω has constant scalar curvature. It is also true that, assuming the topological condition is met, a constant scalar curvature Kähler metric is Kähler-Einstein. This follows from Hodge theory. The Kähler identity $[\Lambda, \partial] = i\bar{\partial}^*$ shows that the scalar curvature S and the Ricci form ρ are related by $\bar{\partial}^*\rho = i\partial S$. Since ρ is also closed, constant scalar curvature implies ρ is harmonic. As ρ and $\lambda\omega$ are two harmonic representatives for the same cohomology class they must be equal. This shows that constant scalar curvature Kähler metrics are a generalisation of Kähler-Einstein metrics which can exist in Kähler classes which are not multiples of the first Chern class.

A great deal of work has been done on determining which manifolds admit Kähler-Einstein metrics. As is remarked above, a necessary condition is that the manifold have definite or zero first Chern class. Aubin's work on the complex Monge-Ampère equations [Aub78] proves that all complex manifolds X with $c_1(X) < 0$ are Kähler-Einstein. In [Yau78], Yau proves the same result for manifolds with $c_1(X) = 0$. The hardest case is $c_1(X) > 0$ (Fano varieties). Examples of such manifolds with no Kähler-Einstein metric are known (*e.g.* $\mathbb{P}^2$ blown up at a point). A large amount of work has been done in this direction by Tian (see, for example, [Tia00]) culminating in a necessary and sufficient condition for a Fano surface to be Kähler-Einstein, namely that its automorphism group be reductive. In higher dimensions this is known to be necessary, but not sufficient. The following section discusses a conjecture which, if true, would give a necessary and sufficient condition for the existence of a constant scalar curvature metric in a given Kähler class.

1.1.4 Stably polarised varieties

Before discussing the relationship between stably polarised varieties and constant scalar curvature Kähler metrics the analogous concepts from the study of holomorphic bundles are described.

A Hermitian structure in a holomorphic bundle $E \to (X, \omega)$ over a compact Kähler manifold is said to be *Hermitian-Einstein* if the curvature F of the induced connection satisfies $F \wedge \omega^{n-1} = C\omega^n$ for some constant C.

Given a coherent sheaf $\mathscr{F}$ on X, the *slope* of $\mathscr{F}$ is defined to be

$$\mu(\mathscr{F}) = \frac{\deg(\mathscr{F})}{\operatorname{rank}(\mathscr{F})} = \frac{1}{\operatorname{rank}(\mathscr{F})} \int_X c_1(\mathscr{F}) \wedge \omega^{n-1}.$$

A holomorphic bundle E is said to be *slope-stable* if, whenever $\mathscr{F}$ is a coherent subsheaf of E, $\mu(\mathscr{F}) < \mu(E)$.

The Hitchin-Kobayashi correspondence for holomorphic vector bundles over a Kähler manifold (X, ω) relates the slope stability of the bundle to the existence of a Hermitian-Einstein structure. More precisely, an indecomposable holomorphic bundle $E \to X$ is slope-stable if, and only if, there exists a Hermitian-Einstein structure in E. This is remarkable since it shows that an *a priori* analytic problem (solving the Hermitian-Einstein PDE) is equivalent to an *a priori* algebro-geometric problem (finding coherent subsheaves and computing their slopes). Over curves, this was proved by Narasimhan and Seshadri [NS65] (see also [Don83] for an alternative proof), over algebraic surfaces by Donaldson [Don85] and over general compact Kähler manifolds by Uhlenbeck and Yau [UY86].

An analogous conjecture has been made for varieties themselves. If (X, L) is a polarised variety, the embedding is conjectured to be "stable" if, and only if, the class $2\pi c_1(L)$ contains a constant scalar curvature Kähler metric. (This conjecture was first stated by Yau [Yau93] in the Fano, Kähler-Einstein case, $L = K_X^*$, by Tian [Tia97] in the constant scalar curvature case, and, later, by Donaldson in the more general framework of symplectic geometry [Don97].) There are competing definitions of stable here. One definition, based on degenerations of (X, L), and called K-stable, is given by Tian in [Tia00] and slightly altered by Donaldson in [Don02b]. An alternative definition, called slope-stable and similar to slope-stability for bundles, has recently appeared in [RT03]. It is known, for example, that K-stability implies slope-stability [RT03] and that the existence of a constant scalar curvature metric implies K-stability (as was recently announced by Chen and Tian).

This conjecture is thought to be the correct generalisation of the correspondence between complex structures on curves and constant scalar curvature metrics mentioned above in 1.1.1. It might be hoped that constant scalar curvature metrics will be of as much use in studying stably polarised varieties as they are in studying Riemann surfaces. In order for this idea to work, and since "most" polarised varieties are stable, most varieties should admit constant scalar curvature metrics. Yet, except for the case of Kähler-Einstein metrics, few existence theorems are known. This thesis is a small attempt to remedy this situation.

1.2 Fibred complex surfaces

This section gives examples of surfaces for which Theorem 1.1 applies. It also gives examples of some particularly simple fibred complex surfaces for which finding constant scalar curvature metrics is straightforward.

1.2.1 Examples of fibred complex surfaces

Fibred complex surfaces with singular fibres are commonplace, *e.g.* holomorphic Lefschetz fibrations. Indeed any algebraic surface is birational to such a surface. The non-singular fibrations dealt with here, however, are harder to find. In [Ati69] and [Kod67] Atiyah and Kodaira independently give examples of such surfaces. Their construction is reviewed here.

Let C be a curve with a fixed-point free, holomorphic, involution $\tau\colon C \to C$. That is, C is an unramified double covering of a curve C'. These exist provided that C' is not the Riemann sphere. To see this, take two copies of C', and cut both along a curve defining a generator of $\pi_1(C')$. Then glue the two copies together so that one side of the cut on one copy is attached to the other side of the cut on the other copy.

In order to ensure that the surface produced is a non-trivial fibre bundle, let the genus g' of C' be at least 2. This means that the genus of C, $g = 2g' - 1$, is at least 3.

Next consider the homomorphism

$$\pi_1(C) \to H_1(C;\mathbb{Z}) \to H_1(C;\mathbb{Z}_2),$$

obtained by abelianising the fundamental group and reducing coefficients modulo 2. Write K for its kernel. Factoring out the universal cover of C by K produces a $4g$-fold covering $f\colon \Sigma \to C$.

The idea is to construct the required surface X as a double cover of $\Sigma \times C$ ramified along the graphs of f and τf. The following lemma will be helpful in this respect.

Lemma 1.2. *Any double cover of C becomes trivial when pulled back to Σ via $f\colon \Sigma \to C$. Equivalently, the induced homomorphism $f_1^*\colon H^1(C;\mathbb{Z}_2) \to H^1(\Sigma;\mathbb{Z}_2)$ is zero.*

Proof. The two statements are equivalent since double covers are classified by H^1 with coefficients in $\mathbb{Z}_2$. A double cover of C corresponds to a homomorphism $\phi\colon \pi_1(C) \to \mathbb{Z}_2$. As $\mathbb{Z}_2$ is abelian, the homomorphism factors through $H_1(C;\mathbb{Z})$. Since the image of K under the map $\pi_1(C) \to H_1(C;\mathbb{Z}_2)$ consists of elements of the form $2a$, $K \subset \ker \phi$. This completes the proof. $\square$

Consider the graphs γ_f and $\gamma_{\tau f}$ of f and τf in $\Sigma \times C$. The following calculation shows that the sum of their homology classes is even.

Lemma 1.3. $[\gamma_f] + [\gamma_{\tau f}] = 0$ *in* $H_2(\Sigma \times C;\mathbb{Z}_2)$.

Proof. It suffices to prove the Poincaré dual is zero in H^2 (working throughout with coefficients in $\mathbb{Z}_2$). Combining the Künneth formula and Poincaré duality gives

$$H^2(\Sigma \times C) \cong \bigoplus \operatorname{Hom}\left(H^i(C), H^i(\Sigma)\right).$$

Under this isomorphism, $\mathrm{PD}[\gamma_f] = \sum f_i^*$ where f_i^* is the homomorphism $H^i(C) \to H^i(\Sigma)$ induced by f. Certainly $f_0^* = 1$ and $f_2^* = \deg f (\bmod 2) = 0$. Similarly $(\tau f)_0^* = 1$ and $(\tau f)_2^* = 0$. It follows from the previous result that $f_1^* = 0$. Also $(\tau f)_1^* = f_1^* \tau_1^* = 0$ proving the lemma. $\square$

This lemma implies the existence of the required double cover $X \to \Sigma \times C$ ramified along γ_f and $\gamma_{\tau f}$. One way to see this is to consider the line bundle L associated to the divisor $\gamma_f + \gamma_{\tau f}$. Since the divisor is even, there exists a line bundle L' such that $L'^2 = L$. L admits a section which vanishes precisely along $\gamma_f + \gamma_{\tau f}$. Define X to be the inverse image of this section under the squaring map $L' \to L$.

The composite holomorphic projection

$$X \to \Sigma \times C \to \Sigma,$$

shows that X is a complex surface of the required type. The fibre over σ is the double cover of C ramified at $f(\sigma)$ and $\tau f(\sigma)$ and so is a non-singular curve with genus twice that of C. From this description it is clear that the complex structures of the fibres are varying. Hence X is not biholomorphic to the product of two curves, at least not in a way that preserves the projection to Σ. (This follows from the fact that $g > 1$ so the group of biholomorphisms of C is not transitive.)

One of the original motivations for Atiyah and Kodaira's construction was to provide an example of a fibre bundle with non zero signature. As is noted in [Ati69], the signature of a fibred complex surface and the amount by which the moduli of the fibres vary are closely related. This is discussed below. In particular the varying moduli of the fibres of X implies X has strictly positive signature. Since the signature of a product of curves is zero, this means that X is not even diffeomorphic (or even homotopic) to a product of curves. It provides an interesting example of the surfaces considered in this thesis.

1.2.2 The signature of fibred complex surfaces

Let $\pi\colon X \to \Sigma$ be a surjective holomorphic submersion from a compact complex surface to a curve, with fibres of genus $g \geq 2$. Hirzebruch's signature formula says that the signature of X is given by $p_1(X)/3$. Calculating the total Pontrjagin class gives

$$p(X) = p(V) \cdot \pi^* p(\Sigma) = 1 + c_1(V)^2.$$

where V is the vertical tangent bundle. So the signature is given by

$$\tau(X) = \frac{1}{3}\int_X c_1(V)^2 = \frac{1}{3}\int_\Sigma \pi_*\left(c_1(V)^2\right).$$

The class $\pi_*(c_1(V)^2)$ has an interpretation in terms of the geometry of the moduli of curves. The surface X determines a map $f\colon \Sigma \to \mathscr{M}$ to the moduli space of curves of genus g. [HM98] describes certain tautological classes α_i on $\mathscr{M}$. The class $\alpha_i \in H^{2i}(\mathscr{M};\mathbb{Z})$ is essentially $\pi_*(c_1(\mathscr{V})^{i+1})$ where $\mathscr{V}$ is the vertical tangent bundle of the universal curve (the problem with this being that the universal curve doesn't exist, at least not as a smooth fibre bundle over $\mathscr{M}$). From this it follows that $\pi^*(c_1(V)^2) = f^*\alpha_1$ and hence,

$$\tau(X) = \frac{1}{3}\langle \alpha_1, f_*[\Sigma]\rangle.$$

The class α_1 is well known to be ample (a fact proved, for example, in [HM98]). The above discussion shows that this can be restated in terms of the signature of X as follows:

Theorem 1.4. *Let $\pi\colon X \to \Sigma$ be a compact connected complex surface admitting a holomorphic submersion to a curve, with fibres of genus at least 2. Then the class $\pi_*(c_1(V)^2)$ is the pull back of an ample class from the moduli space of curves, via the natural map $\Sigma \to \mathscr{M}$.*

Equivalently, the signature of X is non-negative. It is strictly positive if, and only if, the fibres of X have varying moduli.

As far as this thesis is concerned, this result is not only used for showing that Atiyah and Kodaira's construction produces nontrivial fibre bundles. It will prove essential in the analysis in later sections.

1.2.3 Ruled manifolds

A *ruled complex manifold* X is one admitting a proper surjective holomorphic submersion $X \to B$ for which the fibres are all projective spaces. For such manifolds, a theorem analogous to Theorem 1.1 has been proved by Hong in [Hon98] and [Hon99].

If $\pi\colon E \to B$ is a holomorphic bundle over a smooth projective variety, projectivising gives a ruled manifold $\mathbb{P}(E)$. The pullback $\pi^*E \to \mathbb{P}(E)$ has a tautologically defined rank one subbundle. The dual of this bundle, denoted H, is positive when restricted to the fibres of $\mathbb{P}(E)$. Adding a large enough multiple of a positive bundle pulled up from the base gives a positive holomorphic line bundle on $\mathbb{P}(E)$ showing that it is projective. If the base is only Kähler, this procedure still defines Kähler classes on $\mathbb{P}(E)$. They have the form

$$\kappa_r = 2\pi c_1(H) + r\kappa_B$$

for large positive real r, where κ_B is a Kähler class on the base. The precise result proved in [Hon98], [Hon99] is:

Theorem 1.5 (Hong). *Let $E \to B$ be a simple holomorphic vector bundle over a compact complex manifold B. Assume that B admits a constant scalar curvature Kähler metric ω and that E has a Hermitian metric which is Einstein with respect to ω. Assume, moreover, that there are no nontrivial deformations of ω through cohomologous constant scalar curvature metrics.*

Then, for all large r, the Kähler classes on $\mathbb{P}(E)$

$$\kappa_r = 2\pi c_1(H) + r[\omega]$$

contain constant scalar curvature Kähler metrics.

The proof of Hong's theorem is similar to that used here to prove Theorem 1.1. The differences are discussed below. In the case of manifolds ruled over curves (in particular ruled surfaces), however, a straightforward proof can be given. The following argument is from [BDB88].

Theorem 1.6. *Let $E \to \Sigma$ be a holomorphic bundle over a curve with a flat Hermitian metric. Then $\mathbb{P}(E)$ admits a constant scalar curvature Kähler metric.*

Remark. To compare this with the previous result note that over a curve the Einstein condition on a degree zero bundle is equivalent to flatness.

Proof. Since the bundle admits a flat Hermitian metric, it arises from some representation $\pi_1(\Sigma) \to \mathrm{U}(r)$, where E has rank r. This means that $\mathbb{P}(E)$ is the quotient of $\mathbb{P}^{r-1} \times \widetilde{\Sigma}$ by $\pi_1(\Sigma)$, where $\widetilde{\Sigma}$ is the uniformising cover of Σ.

The product $\mathbb{P}^{r-1} \times \widetilde{\Sigma}$ has a natural constant scalar curvature metric. Moreover, $\pi_1(\Sigma)$ acts by isometries with respect to this metric. Hence the metric descends to a metric on $\mathbb{P}(E)$ which also has constant scalar curvature. $\square$

Notice that in the statements of Theorems 1.1 and 1.5, the fibres have constant scalar curvature metrics. The most important difference between the theorems is that in 1.1 the fibres have no nonzero holomorphic vector fields, whilst in 1.5 the fibres have holomorphic vector fields, which are not Killing. If (F, ω) is a compact Kähler manifold with constant scalar curvature Kähler metric, and ξ is a holomorphic vector field on F which is not Killing, then flowing ω along ξ gives a non-trivial family of cohomologous constant scalar curvature Kähler metrics.

From the point of view of the analysis, this situation leads to problems, as is explained later. It is avoided in Theorem 1.5 by the additional assumption that the bundle be simple. This means that, whilst there are holomorphic vector fields on the fibres of $\mathbb{P}(E)$, they are not induced by a global vector field on $\mathbb{P}(E)$.

From the point of view of the algebraic geometry, this lack of automorphisms can be seen as part of the stability condition — objects with "too many automorphisms" are not stable. The hypotheses on E in Theorem 1.5 are equivalent (by the Hitchin-Kobayashi correspondence) to the slope-stability of E. It may be hoped that this, along with the hypothesis that the base is "stable" (*i.e.* has a constant scalar curvature metric) is sufficient to ensure the "stability" of $\mathbb{P}(E)$ in whatever sense is needed by the conjecture described in Section 1.1.4.

Another important difference between Theorems 1.5 and 1.1 is that the fibres of the manifolds in Theorem 1.5 are rigid, whilst the fibres in Theorem 1.1 have moduli. This will be seen to lead to extra considerations in the proof.

1.2.4 Base genus 0 *or* 1

Return now to case of a complex surface X satisfying the hypotheses of Theorem 1.1. In the case where Σ has genus 0 or 1, a direct proof of the theorem can be given. The argument is an adaptation of that used above for manifolds ruled over curves.

Theorem 1.7. *Let X be a compact connected complex surface admitting a holomorphic submersion $\pi\colon X \to \Sigma$ to a surface of genus 0 or 1, for which all the fibres are of genus at least 2. Then X admits constant scalar curvature metrics.*

Proof. Let g denote the fibre genus. Mapping each fibre of π to its Jacobian determines a map $j\colon \Sigma \to \mathscr{A}_g$ where $\mathscr{A}_g$ denotes the moduli space of principally polarised abelian varieties of dimension g. The universal cover of $\mathscr{A}_g$ is the Siegel upper half space $\mathscr{S}_g$ which can be realised as a bounded domain in $\mathbb{C}^N$.

If $g(\Sigma) = 0$, j lifts to a holomorphic map $\mathbb{P}^1 \to \mathscr{S}_g$ which must be constant. If $g(\Sigma) = 1$, j lifts to a holomorphic map $\mathbb{C} \to \mathscr{S}_g$ which must also be constant. In both cases this means that the original map j is constant. By Torelli's theorem, all the fibres of π are biholomorphic.

As the model fibre S of $X \to \Sigma$ has genus at least 2, its group of biholomorphisms Γ is finite. Define a principal Γ bundle $P \to \Sigma$ by setting the fibre over σ to be the group of biholomorphisms from $\pi^{-1}(\sigma)$ to S. Since P is a cover of Σ, it arises from some representation $\pi_1(\Sigma) \to \Gamma$. Using this representation,

$$X = P \times_{\pi_1(\Sigma)} S.$$

In the case $g(\Sigma) = 0$, this gives $X = S \times \mathbb{P}^1$, which clearly admits constant scalar curvature metrics. If $g(\Sigma) = 1$, X is a quotient of $S \times \mathbb{C}$ by $\pi_1(\Sigma)$. The product $S \times \mathbb{C}$ admits a natural constant scalar curvature metric, and with respect to this metric $\pi_1(\Sigma)$ acts by isometries. Hence the metric descends to a metric on X which also has constant scalar curvature. $\square$

1.3 Adiabatic limits

The main analytic technique which will be used to prove Theorem 1.1 is that of an *adiabatic limit*. The word adiabatic comes from the ancient Greek ἀδιάβατος, which describes a boundary which cannot be crossed. Its use in science originated in the study of thermodynamics, where an adiabatic process is one involving no heat transfer. The adiabatic limit was first employed in physics, where it is used to study slowly changing processes by approximating them with ones in which the time variable is "infinitely stretched out."

The more geometric version of the adiabatic limit considered here has been used by mathematicians over the last twenty or so years to study a variety of problems. Schematically, the idea is as follows. The aim is to solve a partial differential equation $\Phi_g(A) = 0$ for some geometric object A defined on a Riemannian manifold (M, g), where M is the total space of a fibre bundle and where the equation involves the metric. For example, M is a four (real) dimensional manifold, A is a connection on some bundle and $\Phi_g(A) = 0$ is the condition that A is anti-self dual with respect to g. Or, similar to the problem studied in this thesis, g is a fixed choice of metric, A a symmetric tensor, and $\Phi_g(A) = 0$ is the condition that $g + A$ gives a metric of constant scalar curvature.

A family of metrics g_r can be defined by stretching the metric by a factor of r in the horizontal directions. The family of metrics leads to a family of equations $\Phi_r(A) = 0$. Formally, setting r equal to infinity gives another equation $\Phi_\infty(A) = 0$ called the *adiabatic limit* of $\Phi_r(A) = 0$. A solution of the limiting equation can be thought of as approximately solving the original equation for large r. Given such a solution, an implicit function theorem argument is used to show that there is a genuine solution to $\Phi_r(A) = 0$ nearby for sufficiently large r.

The idea is, perhaps, best explained with an example. An adiabatic limit is used in the proof of Dostoglou and Salamon's variant of the Atiyah-Floer conjecture [DS94]. The following section discusses the relevant ideas from that paper.

1.3.1 Instantons and holomorphic discs

The exposition of [DS94] given here is necessarily brief. Various objects in the paper (the moduli space of irreducible flat connections over a Riemann surface, instanton and symplectomorphism Floer homology groups) are not defined here as they are not relevant to the main discussion. They are all defined in [DS94] which also contains further references on these topics.

The variation of the Atiyah-Floer conjecture proved by Dostoglou and Salamon begins with a closed Riemann surface S. Given an orientation preserving diffeomorphism $f\colon S \to S$ the mapping cylinder

$$Y_f = \frac{S \times [0,1]}{(s,0) = (f(s),1)}$$

can be constructed. The diffeomorphism f lifts to a bundle automorphism (unique up to gauge transformations) of the non-trivial SO(3)-bundle $P \to S$. Using this automorphism an SO(3)-bundle $Q \to Y_f$ can be defined and the instanton Floer homology groups of the pair (Y_f, Q) can be constructed.

On the other hand, the bundle automorphism of P lifting f induces a symplectomorphism ϕ of $\mathscr{M}$ the moduli space of flat connections in P. Thus the symplectic Floer homology groups of the pair $(\mathscr{M}, \phi)$ can be constructed.

The result proved by Dostoglou and Salamon is that

$$HF_*^{\text{ins}}(Y_f, Q) \cong HF_*^{\text{symp}}(\mathscr{M}, \phi).$$

They do this by showing that, in fact (for a suitable choice of auxiliary data), the two chain complexes involved are isomorphic. The chain groups themselves are easily identified: fixed points of ϕ correspond to flat connections over Y_f. It is in identifying the two differentials that the adiabatic limit is used.

To define the differential in the instanton picture, a metric is chosen on Y_f. The component of the differential linking flat connections A_+ and A_- is then computed by counting instantons in $Q \times \mathbb{R} \to Y_f \times \mathbb{R}$ which converge to $A_\pm$ as the $\mathbb{R}$ coordinate tends to $\pm\infty$.

Thinking of $Y_f \times \mathbb{R}$ as an S-bundle over $S^1 \times \mathbb{R}$, the metric gives a splitting

$$\Omega^2(Y_f \times \mathbb{R}) = \Omega^2_V \oplus \left(\Omega^1_V \otimes \Omega^1_H\right) \oplus \Omega^2_H$$

corresponding to the splitting of the tangent bundle into vertical and horizontal parts (the $S^1 \times \mathbb{R}$ directions are horizontal, the fibre directions vertical). Given a connection A, write $F = F_{VV} + F_{VH} + F_{HH}$ for the decomposition of its curvature under this splitting.

Scaling the metric by r in the horizontal directions doesn't affect the splitting, but does affect the Hodge star. With respect to the scaled metric, the instanton equations for A become

$$F_{VV} = -r^{-1} * F_{HH}, \tag{1.1}$$

$$F_{VH} = - * F_{VH}, \tag{1.2}$$

where $*$ is the Hodge star of the unscaled metric.

In the adiabatic limit $r \to \infty$, equation (1.1) says that A induces flat connections in the fibres. Thus a solution to the adiabatic limit equations is, in part, a family of flat connections in P, defined up to gauge, *i.e.* a map $A\colon I \times \mathbb{R} \to \mathscr{M}$ with $A(0,t) = \phi\,(A(1,t))$.

Equation (1.2) can be interpreted as the Cauchy-Riemann equations for the map A. To see this, first recall that in the presence of a metric on S, there is an identification

$$T_{[B]}\mathscr{M} \cong \mathscr{H}^1_B(S, \mathfrak{g}_P),$$

where B is a flat connection in P and the right hand side denotes B-harmonic 1-forms with values in $\mathfrak{g}_P$. Under this identification the complex structure on $\mathscr{M}$ is given by minus the Hodge star.

To calculate the derivative of the map A at a point (s,t) fix an identification of the fibre of $Y_f \times \mathbb{R}$ over (s,t) with S, and of Q restricted to this fibre with P. Denote by B the fibrewise connection under these identifications. Then

$$T_{(s,t)}\,(I \times \mathbb{R}) \xrightarrow{F_{VH}} \Omega^1(S, \mathfrak{g}_P) \longrightarrow \mathscr{H}^1_B(S, \mathfrak{g}_P)$$

gives the derivative of A at (s,t), where the second map is projection onto the harmonic part. Conceptually, this is because F_{VH} quantifies the non-commutativity of parallel transport in the fibre directions with parallel transport in the horizontal directions, *i.e.* if we use horizontal parallel transport to identify $Q \times \mathbb{R}$ restricted to nearby fibres of $Y_f \times \mathbb{R}$, F_{VH} measures the infinitesimal change in fibre-wise connection. Since the horizontal parallel transport also changes the fibre-wise identifications, it is necessary to project onto the harmonic part of F_{VH} to obtain the infinitesimal change in the connection over (s,t) with respect to the original identification. Alternatively, this can be proved by local calculation.

Still using these identifications, write $F_{VH} = \alpha \otimes \mathrm{d}s + \beta \otimes \mathrm{d}t$ where $\alpha, \beta \in \Omega^1(S, \mathfrak{g}_P)$. Equation (1.2) now reads $*_S\alpha = -\beta$, where $*_S$ is the Hodge star on S. As $DA(\partial_s) = \alpha$, $DA(\partial_t) = \beta$ and $J\partial_s = \partial_t$, and the Hodge star commutes with harmonic projection, this is precisely the condition that A is holomorphic.

The conclusion is that solutions of the adiabatic limit of the instanton equations correspond to the holomorphic strips $I \times \mathbb{R} \to \mathscr{M}$ used to define the differential in the symplectic Floer complex. What Dostoglou and Salamon prove is that this is not just a formal correspondence. It is, for large enough values of r, a bijection.

Each holomorphic strip determines a connection over $Y_f \times \mathbb{R}$. The connection is obtained by pulling back the universal connection in the universal bundle over $\mathscr{M}$. The construction of these universal objects is explained in sections 5.1.1 and 5.2.3 of [DK90]. In the limit $r \to \infty$ the self-dual part of the curvature of this connection (with respect to the r-scaled metric) tends to zero. The implicit function theorem then shows that, for each large r, there is a unique instanton (with respect to the r-scaled metric) nearby. Conversely, for large enough r, every instanton is obtained this way. This shows that the differentials used to compute the different Floer homology groups are equal, proving the result.

1.4 Outline of proof

This section gives an outline of the proof of Theorem 1.1.

The first step is to construct a family of approximate solutions. This is done in Chapter 4. The motivating idea is that in an adiabatic limit the local geometry is dominated by that of the fibre. The approximate solutions are essentially constructed, then, by fitting together the constant scalar curvature metrics on the fibres of X and stretching out the base. If the base is scaled by a factor r, these metrics these metrics have scalar curvature which is $O(r^{-1})$ from being constant. Chapter 4 also discusses adjusting these metrics to decrease the error to $O(r^{-n})$ for any positive integer n.

The remaining chapters carry out the hard work of showing that a genuine solution lies nearby. Doing this involves solving a parameter dependent implicit function theorem. As is explained in Section 5.1, such arguments hinge on certain analytic estimates. In particular the right inverse of the derivative involved must be controlled uniformly with respect to the parameter.

Section 5.2 discusses the example of the product $S \times T^2$ where the flat torus T^2 is scaled by a factor r. By considering the required estimates in this specific case, the general behaviour of the relevant linear operator can be guessed at. Moreover, as is discussed in Chapter 6, this example provides a local model for the metric in the general case. This provides justification for the statement that, as $r \to \infty$, the geometry of the fibre dominates. It also enables the relevant local analytic estimates (Sobolev inequalities, elliptic estimates *etc.*) to be proved.

Controlling the inverse of the derivative is a global analytic problem, and so the local model is not of direct use. Instead, in Chapter 7, a global model is used which is easier to compute with. Finally, Chapter 8 tidies up the remaining loose ends needed to use a parameter dependent implicit function theorem and complete the proof of Theorem 1.1.

Before the start of the proof itself, Chapters 2 and 3 collate some background information which will feature later on. Chapter 2 briefly describes the correspondence between constant curvature metrics and complex structures which holds for Riemann surfaces. In particular it proves that on a high genus compact surface, the hyperbolic metric corresponding to a particular complex structure varies smoothly with the complex structure. Chapter 3 proves some straightforward properties of the scalar curvature and its dependence on the Kähler structure. In particular it states precisely how various geometric objects are uniformly continuous with respect to the metric used to define them. The results of both these chapters are standard and widely known, although not stated in the literature in quite the form used here.

2 *Constant curvature metrics on curves*

This chapter discusses the classical correspondence on high genus compact Riemann surfaces, between complex structures and constant scalar curvature metrics. The existence and uniqueness of such metrics is deduced from an existence and uniqueness theorem for a certain partial differential equation. This theorem is used again in a different, but related, context in Chapter 4.

The results in this chapter are well known. [Tro92] is a general reference for such things, whilst [KW71] takes the point of view explained here. In particular, the equation (2.4) is discussed there.

2.1 Complex structures and conformal classes

This section describes briefly the correspondence, on oriented smooth surfaces, between conformal classes and complex structures. Recall that a diffeomorphism $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if it is conformal. This means that any holomorphic coordinate chart on a Riemann surface S gives S a natural conformal structure.

The converse result is more difficult. Given a conformal class on S, choose a representative metric g. It follows essentially from the Riemann mapping theorem that there are local coordinates (x, y) on S in which g has the form

$$g = F(x, y)\left(\mathrm{d}x^2 + \mathrm{d}y^2\right).$$

Such coordinates are called *isothermal*. For a short proof of the existence of isothermal coordinates see [Che55]. Oriented changes of coordinates between isothermal patches are conformal, hence holomorphic. This gives a one-to-one correspondence between complex structures and conformal classes.

2.2 Conformal classes and constant curvature metrics

This section proves that, on an oriented surface with genus at least 2 and of fixed area, any conformal class contains a unique constant scalar curvature metric. It also proves that the corresponding metric depends smoothly on the conformal class. Combined with the correspondence described in the previous section this gives a one-to-one correspondence between complex structures and constant scalar curvature metrics of prescribed area.

2.2.1 Existence and uniqueness

Let S be an oriented surface and $[g]$ a conformal class on S. Any representative for the class can be written as $g' = e^{\phi}g$ for some function ϕ. The conformal class $[g]$ determines a complex structure on S. Using this, the Ricci forms ρ and ρ' of the metrics g and g' are related by

$$\rho' = \rho + i\bar{\partial}\partial\phi. \tag{2.1}$$

Denoting the trace operators corresponding to g and g' by Λ and Λ' respectively we have

$$\Lambda' = e^{-\phi}\Lambda. \tag{2.2}$$

Equations (2.1) and (2.2) show that the scalar curvatures Scal and Scal$'$ of the two metrics are related by

$$\mathrm{Scal}'\, e^{\phi} = \mathrm{Scal} + \Delta\phi, \tag{2.3}$$

where Δ is the g-Laplacian.

By rescaling, if a given conformal class admits a constant scalar curvature metric, it admits one with scalar curvature minus one. To show the existence of a unique solution ϕ to equation (2.3) with $\mathrm{Scal}' = -1$ it is sufficient to show that there is a unique solution $\phi \in C^\infty$ to the partial differential equation

$$\Delta\phi + e^{\phi} = \psi, \tag{2.4}$$

for any $\psi \in C^\infty$ such that $\int_S \psi > 0$. (This holds for $\psi = -\mathrm{Scal}$ by Gauss-Bonnet.)

First, a lemma is proved that will establish uniqueness and *a priori* bounds on solutions of (2.4).

Lemma 2.1. *Suppose $\phi_\pm \in C^2$ are functions satisfying*

$$\begin{aligned} \Delta\phi_+ + e^{\phi_+} &> \psi, \\ \Delta\phi_- + e^{\phi_-} &< \psi, \end{aligned}$$

and $\phi \in C^2$ is a solution of

$$\Delta\phi + e^{\phi} = \psi.$$

Then $\phi_- < \phi < \phi_+$. The lemma remains true with strict inequalities replaced throughout by weak ones.

Proof. Put $\chi = \phi_+ - \phi$. Notice that $\Delta\chi + e^{\phi}(e^{\chi} - 1) > 0$. As S is compact, χ attains its minimum. At such points $\Delta\chi \leq 0$. Thus $e^{\chi} > 1$ at the minima of χ and hence everywhere. Therefore $\phi < \phi_+$. The proofs of $\phi_- < \phi$ and the weak inequalities version are similar. $\square$

Next, a lemma is proved on regularity of solutions of (2.4).

Lemma 2.2. *Let $\psi \in L^2_k$ for $k \geq 2$ and suppose that $\phi \in L^2_2$ solves*

$$\Delta\phi + e^{\phi} = \psi.$$

Then $\phi \in L^2_{k+2}$. In particular if $\psi \in C^\infty$ then $\phi \in C^\infty$.

Proof. For $l \geq 2$, exponentiation defines a map $L^2_l \to L^2_l$. To see this note that on a Riemann surface $L^2_l \hookrightarrow C^0$ for $l \geq 2$. This implies products of functions in L^2_l are also in L^2_l, and hence analytic operations (in particular $\phi \mapsto e^{\phi}$) map $L^2_l \to L^2_l$.

This means that $\Delta\phi = \psi - e^{\phi}$ is in L^2_2. Elliptic regularity for the Laplacian implies $\phi \in L^2_4$. Iterating the argument gives the result. □

Theorem 2.3. *Let S be a compact oriented surface of genus at least 2 with a Riemannian metric. If $\psi \in L^2_2$ has strictly positive integral then there exists a unique solution $\phi \in L^2_4$ to the equation*

$$\Delta\phi + e^{\phi} = \psi.$$

Proof. Lemma 2.1 gives uniqueness: if ϕ and ϕ' are solutions of (2.4) put $\phi_- = \phi' = \phi_+$ in the weak inequalities version to conclude that $\phi = \phi'$. (The Sobolev embedding $L^2_4 \hookrightarrow C^2$ ensures $\phi, \phi' \in C^2$.)

A continuity method is used to prove existence. Let A denote the set of functions $\psi \in L^2_2$ with strictly positive integral, and $A' \subset S$ denote the subset of functions for which (2.4) has a solution $\phi \in L^2_4$. A is a convex set and so is connected. When $\psi = 1$, $\phi = 0$ solves (2.4) so A' is non-empty. It suffices, then, to show that S' is both open and closed in A (in the L^2_2-topology).

A' is open

Define a map $F\colon L^2_4 \to L^2_2$ by

$$F(\phi) = \Delta\phi + e^{\phi}.$$

F is differentiable, its derivative at ϕ being given by

$$DF_\phi(\chi) = \Delta\chi + e^{\phi}\chi.$$

This is an elliptic, self adjoint, strictly positive operator. By the Fredholm alternative it is an isomorphism.

By the inverse function theorem, each point $\psi \in A'$ has a neighbourhood on which F is invertible. That is, A' is open.

A' is closed

Take a sequence (ψ_n) in A' converging in L^2_2 to a function $\psi \in S$. Denote by $\phi_n \in L^2_4$ the functions satisfying

$$\Delta \phi_n + e^{\phi_n} = \psi_n.$$

To show $\psi \in A'$ a solution of (2.4) must be found.

Choose functions $\phi_\pm$ as in Lemma 2.1 (with strict inequalities). For ϕ_+ take any constant bigger than $\|\log \psi\|_{C^0}$. To find ϕ_- let $\bar{\psi} = \int_S \psi$ denote the average value of ψ. As $\psi - \bar{\psi}$ is L^2-orthogonal to the constants there exists ϕ_- such that $\Delta \phi_- = \psi - \bar{\psi}$. By regularity for the Laplacian, $\phi_- \in L^2_4$. Hence, by Sobolev embedding, $\phi_- \in C^2$. This choice of ϕ_- works unless $e^{\phi_-} \geq \bar{\psi}$ at some point of S. Since $\bar{\psi} > 0$ this can be prevented by subtracting a large constant from ϕ_-.

By Sobolev embedding, $\psi_n \to \psi$ in C^0. So for sufficiently large n,

$$\begin{aligned} \Delta \phi_+ + e^{\phi_+} &> \psi_n, \\ \Delta \phi_- + e^{\phi_-} &< \psi_n. \end{aligned}$$

Hence $\phi_- < \phi_n < \phi_+$. This gives an *a priori* estimate for ϕ_n, *i.e.* $\|\phi_n\|_{C^0}$ is bounded.

This estimate can be improved to an L^2_4 bound. The C^0 bound on ϕ_n implies that e^{ϕ_n} is bounded in C^0, and hence also in L^2. As ψ_n converges in L^2_2 it is bounded in L^2_2, and so in L^2. Since

$$\Delta \phi_n = \psi_n - e^{\phi_n},$$

it follows that $\Delta \phi_n$ is bounded in L^2. As ϕ_n has already be shown to be bounded in L^2 (via C^0), ϕ_n is bounded in L^2_2.

The map $\phi \mapsto e^\phi$ is continuous as a map on L^2_2. This implies that e^{ϕ_n} is bounded in L^2_2. Hence $\Delta \phi_n$ is bounded in L^2_2, and so ϕ_n is bounded in L^2_4 as required.

The embedding $L^2_4 \hookrightarrow C^2$ is compact, so ϕ_n has an C^2-convergent subsequence. Its limit ϕ is a C^2 solution of (2.4). By Lemma 2.2, $\phi \in L^2_4$. This proves A' is closed. $\square$

This theorem, together with Lemma 2.2, proves the following.

Theorem 2.4. *On a compact, oriented surface of genus at least two, each conformal class contains a unique metric with scalar curvature identically minus one.*

2.2.2 Smooth dependence

A family of metrics $\{g_t : t \in \mathbb{R}\}$ on S is said to be smooth if it corresponds to a smooth tensor over $S \times \mathbb{R}$.

Proposition 2.5. *Let $\{g_t : t \in \mathbb{R}\}$ be a smooth family of metrics on a compact orientable surface of genus $g \geq 2$. Let ϕ_t denote the corresponding functions such that $\mathrm{e}^{\phi_t} g_t$ has constant scalar curvature -1. Then ϕ depends smoothly on t.*

Proof. Let Scal_t be the scalar curvature of g_t. Then ϕ_t is the unique solution of the equation

$$\Delta\phi_t + e^{\phi_t} = -\,\mathrm{Scal}_t\,.$$

The formula for scalar curvature shows that, since g_t depends smoothly on t, so does S_t. Since the surface is compact, the C^0-topology is stronger than the L^2-topology. Hence $t \mapsto \mathrm{Scal}_t$ defines a smooth map of Banach spaces $\mathbb{R} \to L^2$.

The inverse function theorem (as used to prove A' is open during the proof of Theorem 2.3) gives that $t \mapsto \phi_t$ is a smooth map of Banach spaces $\mathbb{R} \to L^2_2$. By Sobolev embedding, the L^2_2 topology is stronger than the C^0 topology. Hence ϕ_t depends smoothly on t.

As an expansion of this argument, the existence of the first derivative is proved in detail. Since the surface is compact,

$$|h|^{-1} \left\| \mathrm{Scal}_{t+h} - \mathrm{Scal}_t - h\frac{\partial\,\mathrm{Scal}}{\partial t} \right\|_{L^2}$$

is bounded above by

$$\mathrm{const.}|h|^{-1} \left\| \mathrm{Scal}_{t+h} - \mathrm{Scal}_t - h\frac{\partial\,\mathrm{Scal}}{\partial t} \right\|_{C^0}.$$

The second expresion, and hence the first, tends to zero as $h \to 0$. So $t \mapsto \mathrm{Scal}_t$ is a differentiable map of Banach spaces $\mathbb{R} \to L^2$.

By the inverse function theorem, $t \mapsto \phi_t$ is a differentiable map of Banach spaces. Its derivative at t is a bounded linear map $D\phi_t \colon \mathbb{R} \to L^2_2$. By Sobolev embedding,

$$|h|^{-1} \|\phi_{t+h} - \phi_t - hD\phi_t(1)\|_{C^0} \leq \text{const.}|h|^{-1} \|\phi_{t+h} - \phi_t - D\phi_t(h)\|_{L^2_2}.$$

The right hand side tends to zero as $h \to 0$, so ϕ_t is differentiable with respect to t (with $\partial\phi/\partial t = D\phi_t(1)$). $\square$

General properties of scalar curvature *3*

This chapter details some general properties of the scalar curvature of Kähler manifolds. Parts of the proof of Theorem 1.1 involve comparing the actual metric at hand to a local model. Correspondingly, part of this chapter concentrates on comparing the scalar curvatures of different Kähler structures on the same underlying manifold.

3.1 The scalar curvature map

Kähler metrics on a compact Kähler manifold X in a fixed cohomology class are parametrised by Kähler potentials: any other Kähler form cohomologous to a given Kähler metric ω is of the form

$$\omega_\phi = \omega + i\bar{\partial}\partial\phi \tag{3.1}$$

for some real valued $\phi \in C^\infty$. (This follows from the $\bar{\partial}\partial$-Lemma.) Equation (3.1) determines ϕ up to an additive constant. Moreover, equation (3.1) defines a Kähler metric providing the second derivatives of ϕ satisfy an open condition corresponding to the condition that ω_ϕ be positive. In this way the Kähler metrics in the class $[\omega]$ are parametrised by an open set $U \subset C^\infty$ modulo the action of $\mathbb{R}$ acting by addition.

The scalar curvature of a Kähler metric can be expressed as the trace of the Ricci form with respect to the Kähler form:

$$\mathrm{Scal}(\omega)\,\omega^n = n\rho \wedge \omega^{n-1},$$

where n is the complex dimension of X. Scalar curvature defines a map $S : U \to C^\infty$, $S(\phi) = \mathrm{Scal}(\omega_\phi)$.

The equation studied in this thesis is

$$\mathrm{Scal}(\omega_\phi) = \mathrm{const}.$$

This is a fourth order fully nonlinear nonlinear partial differential equation for ϕ. (Fully nonlinear means that the nonlinearities involve the highest order derivatives of ϕ.)

Lemma 3.1. *Let V denote the L^p_{k+4}-completion of U. S extends to a map $S : V \to L^p_k$ whenever $(k+2)p - 2n > 0$.*

Proof. The condition on k and p ensures that the Sobolev embedding theorem applies (see,*e.g.* [Aub82]); so that $L^p_{k+2} \hookrightarrow C^0$. In particular, for $\phi \in L^p_{k+4}$, the metric $\omega_\phi \in L^p_{k+2}$ is continuous. Locally, the Ricci form is given by $\rho = i\bar{\partial}\partial \log\det g$, where g is the metric on the tangent bundle. Since $\log\det g$ depends analytically on g and since g is continuous, $\log\det g \in L^p_{k+2}$ also. It follows that $\rho \in L^p_k$. Finally, since taking the trace with respect to a continuous metric defines a map $L^p_k \to L^p_k$, $S(\phi) \in L^p_k$. $\square$

Lemma 3.2. *The map $S : V \to L^p_k$ is smooth. Its derivative at the origin is given by*

$$L(\phi) = \left(\Delta^2 - S(0)\Delta\right)\phi + n(n-1)\frac{i\bar{\partial}\partial\phi \wedge \rho \wedge \omega^{n-2}}{\omega^n}. \tag{3.2}$$

Proof. The proof that S extends to a map between Sobolev spaces shows, in fact, that $S(\phi)$ depends analytically on ϕ and its derivatives up to fourth order. In particular S is smooth.

To compute its derivative, let $\omega_t = \omega + ti\bar{\partial}\partial\phi$. The corresponding metric on the tangent bundle is $g_t = g + t\Phi$ where Φ is the real symmetric tensor corresponding to $i\bar{\partial}\partial\phi$. The Ricci form is given locally by

$$\rho_t = \rho + i\bar{\partial}\partial \log\det\left(1 + tg^{-1}\Phi\right).$$

Now $\operatorname{tr}(g^{-1}\Phi) = \Lambda(i\bar{\partial}\partial\phi) = \Delta\phi$. Hence, at $t = 0$,

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = i\bar{\partial}\partial\phi, \qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} = i\bar{\partial}\partial(\Delta\phi).$$

The result follows from differentiating the equation $S\omega^n = n\rho \wedge \omega^{n-1}$. $\square$

Remarks 3.3.

1. In flat space $L = \Delta^2$.

2. In general, the leading order term of L is Δ^2. It follows immediately that L is elliptic, with index zero.

3. It follows, either from the formula for L, or from the fact that $\int S(\phi)\omega_\phi^n$ is constant, that

   $$\int L(\phi)\,\omega^n = -\int \phi\Delta S(0)\,\omega^n.$$

   In particular, for metrics of constant scalar curvature, $\operatorname{im} L$ is L^2-orthogonal to the constant functions.

4. By symmetry, the derivative of S at a point $\psi \in L^p_{k+4}$ is given by a similar expression where all the quantities are calculated with respect to the metric ω_ψ.

Example 3.4 (High genus curves). For the hyperbolic metric on a high genus curve, $S(0) = -1$. Hence the above lemma gives

$$L = \Delta^2 + \Delta.$$

As mentioned above, L has index 0. If $\phi \in \ker L$ then, by standard elliptic regularity arguments (see, *e.g.* [Aub82]), ϕ is smooth. Moreover,

$$0 = \langle \phi, L\phi \rangle = \|\Delta\phi\|_{L^2}^2 + \|\mathrm{d}\phi\|_{L^2}^2.$$

Hence ϕ is constant. Considered as a map between spaces of functions with mean value zero, L is an isomorphism.

Since Theorem 1.1 is concerned with a family of high genus curves, this result will be useful in several places in this thesis.

Example 3.5 (Kähler products). Let (F, ω_F) and (B, ω_B) be Kähler manifolds with L_F and L_B denoting the corresponding linearisations of scalar curvature, and Laplacians Δ_F and Δ_B. Let $X = F \times B$ and consider the Kähler metric

$$\omega_r = \omega_F \oplus r\omega_B.$$

Direct calculation using formula (3.2) shows that the linearisation of the scalar curvature map defined by ω_r is

$$L_F + 2r^{-1}\Delta_F\Delta_B + r^{-2}L_B.$$

This formula will be useful in section 5.2.

In the above discussion of scalar curvature, the underlying complex manifold (X, J) is regarded as fixed, whilst the Kähler form ω is varying. An alternative point of view is described in [Don97]. There the symplectic manifold (X, ω) is fixed, whilst the complex structure is varying (through complex structures compatible with ω). The two points of view are related as follows.

Calculation shows that, on a Kähler manifold (X, J, ω),

$$2i\bar{\partial}\partial\phi = \mathscr{L}_{\nabla\phi}\omega.$$

Hence the change in ω due to the Kähler potential ϕ is precisely that caused by flowing ω along $\nabla\phi$. Flowing the Kähler structure back along $-\nabla\phi$ restores the original symplectic form, but changes the complex structure by

$-\mathscr{L}_{\nabla\phi}J$. This means that the two points of view (varying ω versus varying J) are related by the diffeomorphism generated by $\nabla\phi$.

The following result, on the first order variation of scalar curvature under changes in complex structure, is proved in [Don97]. The operator

$$\mathscr{D} : C^\infty(X) \to \Omega^{0,1}(TX)$$

is defined by $\mathscr{D}\phi = \bar\partial\nabla\phi$, where $\bar\partial$ is the $\bar\partial$-operator of the holomorphic tangent bundle. The operator $\mathscr{D}^*$ is the formal adjoint of $\mathscr{D}$ with respect to the L^2-inner product determined by the Kähler metric.

Lemma 3.6. *An infinitesimal change of $-\mathscr{L}_{\nabla\phi}$ in the complex structure J causes an infinitesimal change of $\mathscr{D}^*\mathscr{D}\phi$ in the scalar curvature of (X, J, ω).*

Taking into account the diffeomorphism required to relate this point of view to that in which ω varies gives the following formula for the linearisation of scalar curvature with respect to Kähler potentials:

Lemma 3.7.

$$L(\phi) = \mathscr{D}^*\mathscr{D}\phi + \nabla \operatorname{Scal} \cdot \nabla\phi \tag{3.3}$$

If the scalar curvature is constant, then $L = \mathscr{D}^*\mathscr{D}$. In particular $\ker L = \ker \mathscr{D}$ consists of functions with holomorphic gradient. If X has constant scalar curvature and no holomorphic vector fields, then $\ker L = \mathbb{R}$. Since L is also self adjoint, L is an isomorphism between spaces of functions with mean value zero. This generalises Example 3.4 (which considered a high genus curve with its hyperbolic metric).

3.2 Dependence on the Kähler structure

This section proves that the scalar curvature map is uniformly continuous under changes of the Kähler structure. The arguments are straightforward, this section simply serves to give a precise statement of estimates that will be used later.

3.2.1 C^k-topology

The results are proved first using the C^k-topology. The Leibniz law implies that there is a constant C such that for tensors T, $T' \in C^k$

$$\|T \cdot T'\|_{C^k} \leq C\|T\|_{C^k}\|T'\|_{C^k}. \tag{3.4}$$

The dot stands for any algebraic operation involving tensor product and contraction. The constant C depends only on k, not on the metric used to calculated the norms (in contrast to the Sobolev analogue, which is discussed in the next section).

Lemma 3.8. *There exist positive constants c, K, such that whenever g, g' are two different metrics on the same compact manifold, satisfying*

$$\|g' - g\|_{C^{k+2}} \leq c,$$

with corresponding curvature tensors R, R', then

$$\|R' - R\|_{C^k} \leq K\|g' - g\|_{C^{k+2}}.$$

All norms are taken with respect to the metric g'. K depends only on c and k (and not on g or g').

Proof. Let $g = g' + h$. If the corresponding Levi-Civita connections are denoted ∇ and ∇', then $\nabla = \nabla' + a$, where a corresponds to $-\nabla' h$ under the isomorphism $T^* \otimes \operatorname{End} T \cong T^* \otimes T^* \otimes T^*$ defined by g. That is,

$$a \cdot (g' + h) = -\nabla' h,$$

where the dot denotes some algebraic operation. Hence

$$\begin{aligned} \|a\|_{C^{k+1}} &= \|a \cdot g'\|_{C^{k+1}}, \\ &\leq \|\nabla' h\|_{C^{k+1}} + C\|a\|_{C^{k+1}}\|h\|_{C^{k+1}}, \end{aligned}$$

(using inequality (3.4) above). Taking $c < C^{-1}$ gives

$$\|a\|_{C^{k+1}} \leq \|h\|_{C^{k+2}} \left(1 - c^{-1}\|h\|_{C^{k+2}}\right)^{-1}.$$

The difference in curvatures is given by

$$R - R' = \nabla' a + \frac{1}{2} a \wedge a.$$

The result now follows from (3.4). $\square$

Remark. For uniformity, it is essential that the norms are measured with respect to one of the metrics involved. If the norms were taken with respect to a third metric g'', the bound would also depend on $\|g' - g''\|_{C^k(g'')}$.

Lemma 3.9. *Given k and $M > 0$, there exist positive constants c and K such that whenever g and g' are two different metrics on the same compact manifold, satisfying*

$$\begin{aligned} \|g' - g\|_{C^{k+2}} &\leq c, \\ \|R'\|_{C^k} &\leq M, \end{aligned}$$

where R' is the curvature tensor of g', then

$$\| \operatorname{Ric}' - \operatorname{Ric} \|_{C^k} \leq K \|g' - g\|_{C^{k+2}}.$$

Here Ric *and* Ric$'$ *are the Ricci tensors of g and g' respectively and all norms are taken with respect to the metric g'.*

Proof. The Ricci tensor is given by $\operatorname{Ric} = R \cdot g$ where the dot denotes contraction with the metric. Simple algebra gives

$$\operatorname{Ric}' - \operatorname{Ric} = (R' - R) \cdot g' - (R' - R) \cdot (g' - g) + R' \cdot (g' - g).$$

It follows from inequality (3.4) that $\| \operatorname{Ric}' - \operatorname{Ric} \|_{C^k}$ is controlled by a constant multiple

$$\|R' - R\|_{C^k} \|g'\|_{C^k} + \|R' - R\|_{C^k} \|g' - g\|_{C^k} + \|R'\|_{C^k} \|g' - g\|_{C^k}.$$

Since the C^k-norm of g' is constant, the result follows from Lemma 3.8. $\square$

Lemma 3.10. *Given k and $M > 0$, there exist positive constants c and K such that whenever g and g' are two different metrics on the same compact manifold, satisfying*

$$\begin{aligned} \|g' - g\|_{C^{k+2}} &\leq c, \\ \|R'\|_{C^k} &\leq M, \end{aligned}$$

where R' is the curvature tensor of g', then

$$\| \operatorname{Scal}' - \operatorname{Scal} \|_{C^k} \leq K \|g' - g\|_{C^{k+2}}.$$

Here Scal *and* Scal$'$ *are the scalar curvatures of g and g' respectively and all norms are taken with respect to the metric g'.*

Proof. As $\operatorname{Scal} = \operatorname{Ric} \cdot g$, the argument is similar to the proof of Lemma 3.9. $\square$

Returning to the Kähler case, these lemmas show that the derivative of the scalar curvature map (described in the previous section) is uniformly continuous with respect to the Kähler structure used to define it.

Lemma 3.11. *Given k and $M > 0$, there exist positive constants c and K such that whenever (J, ω), (J', ω') are two different Kähler structures on the same compact manifold satisfying*

$$\begin{aligned} \|(J', \omega') - (J, \omega)\|_{C^{k+2}} &\leq c, \\ \|R'\|_{C^k} &\leq M, \end{aligned}$$

where R' is the curvature tensor of (J', ω'), then the linearisations L and L' of the corresponding scalar curvature maps satisfy

$$\big\|(L' - L)(\phi)\big\|_{L^p_k} \leq K \|(J', \omega') - (J, \omega)\|_{C^{k+2}} \|\phi\|_{L^p_{k+4}}.$$

All norms are computed with respect to the primed Kähler structure.

Proof. The formula (3.2) shows that L is a sum of compositions of the operators Δ, $i\bar{\partial}\partial$, multiplication by Scal, ρ and ω, and division of top degree forms by ω^n. It suffices, then, to show that these operations satisfy inequalities analogous to that in the statement of the lemma.

For multiplication by ω this is immediate. Since dividing top degree forms by ω^n is the same as taking the inner product with $\omega^n/n!$ it holds for this operation too.

For multiplication by Scal and ρ, the inequalities follow from Lemmas 3.9 and 3.10 and the inequality

$$\|uv\|_{L^p_k} \leq C\|u\|_{C^k}\|v\|_{L^p_k}$$

for some C (depending only on k).

On functions, Δ is the trace of $i\bar{\partial}\partial$, so to prove the lemma it suffices to prove that $i\bar{\partial}\partial$ satisfies the relevant inequality (*c.f.* the proof of Lemma 3.9). Since

$$\pi^{1,0} = \frac{1}{2}(1 - iJ),$$

the operator $\partial = \pi^{1,0}\mathrm{d}$ satisfies the required inequality. Similarly for $\bar{\partial}$. Hence $i\bar{\partial}\partial$ and Δ do too.

Putting this all together proves the result. ☐

Of course, it is also be possible to prove this result using equation (3.3) in place of (3.2).

3.2.2 L^p_k-topology

In the above discussion of continuity, it is possible to work with Sobolev rather than C^k norms. The same arguments apply with one minor modification. Inequality (3.4) is replaced by

$$\|T \cdot T'\|_{L^p_k} \leq C\|T\|_{L^p_k}\|T'\|_{L^p_k},$$

which holds provided $L^p_k \hookrightarrow C^0$. Moreover, the constant C depends on the metric through the constants appearing in the Sobolev inequalities

$$\begin{aligned} \|S\|_{C^0} &\leq C'\|S\|_{L^p_k} \quad \text{for } kp > 2n, && (3.5)\\ \|S\|_{L^p_k} &\leq C''\|S\|_{L^q} \quad \text{for } kp > 2n. && (3.6) \end{aligned}$$

With this in mind, the same chain of reasoning which leads to Lemma 3.11 also proves:

Lemma 3.12. *Let k, p and n satisfy $kp-2n > 0$, and M be a positive constant. There exist positive constants c and K such that if (J,ω) (J',ω') are two different Kähler structures on the same complex n-dimensional manifold satisfying*

$$\begin{aligned} \|(J',\omega') - (J,\omega)\|_{L^p_{k+2}} &\leq c,\\ \|R'\|_{L^p_k},\, C',\, C'' &\leq M, \end{aligned}$$

where R' is the full curvature tensor and C', C'' are the Sobolev constants from inequalities (3.5) and (3.6) for the primed Kähler structure, then the linearisations L and L' of the corresponding scalar curvature maps satisfy

$$\big\|(L'-L)(\phi)\big\|_{L^p_k} \leq K\|(J',\omega')-(J,\omega)\|_{L^p_{k+2}}\|\phi\|_{L^p_{k+4}}.$$

All norms are computed with respect to the primed Kähler structure.

Approximate solutions 4

Throughout the remainder of this thesis, X is a compact connected complex surface and $\pi\colon X \to \Sigma$ is a holomorphic submersion onto a smooth curve with fibres of genus at least 2.

This chapter constructs families of metrics on X each depending on a parameter r. As $r \to \infty$, the scalar curvature of these metrics approaches minus one. As will be seen, increasing r can be thought of as "stretching out the base."

During this chapter, various power series expansions in negative powers of r will be used. At this stage, the calculations are meant purely formally. Questions of convergence with respect to various Banach space norms will be addressed later. The expression $O(r^{-n})$ is also to be interpreted formally, *i.e.* it represents an arbitrary term with a factor of r^{-n-k} for some $k \geq 0$. Alternatively, when used to describe functions, it can be interpreted in terms of pointwise convergence.

The ultimate aim of this chapter is to construct, for each non-negative integer n, a family of metrics $\omega_{r,n}$ parametrised by r, satisfying

$$\mathrm{Scal}(\omega_{r,n}) = -1 + \sum_{i=1}^{n} c_i r^{-i} + O\left(r^{-n-1}\right),$$

where $c_1, \ldots, c_n$ are constants. (The actual value of the constants will also be established; see the end of this chapter.) This is accomplished in Theorem 4.14. As will be seen, constructing $\omega_{r,0}$ and $\omega_{r,1}$ involves slightly different issues to the higher order approximate solutions, all of which can be constructed recursively.

4.1 The first order approximate solution

Recall the cohomology classes

$$\kappa_r = -2\pi\left(c_1(V) + r c_1(\Sigma)\right)$$

mentioned in the introduction. Here V denotes the vertical tangent bundle over X and r is a positive real number.

Lemma 4.1. *For all sufficiently large r, κ_r is a Kähler class. Moreover, it contains a Kähler representative ω_r whose fibrewise restriction is the canonical hyperbolic metric on that fibre.*

Proof. Each fibre has a canonical hyperbolic metric. These metrics vary smoothly from fibre to fibre and so define a Hermitian structure in the holomorphic bundle $V \to X$. (See Chapter 2 for proofs of these statements.) Denote the corresponding curvature form by F_V, and define a closed real $(1,1)$-form by

$$\omega_0 = -iF_V.$$

Notice that $[\omega_0] = -2\pi c_1(V)$.

The fibrewise restriction of F_V is just the curvature of the fibre with respect to its hyperbolic metric. This implies that the restriction of ω_0 to a fibre is the hyperbolic metric itself.

Since the fibrewise restriction of ω_0 is nondegenerate, it defines a splitting $TX = V \oplus H$, where

$$H_x = \{u \in T_xX : \omega_0(u,v) = 0 \quad \text{for all } v \in V_x\}.$$

Let ω_Σ be any Kähler form on the base, scaled so that $[\omega_\Sigma] = -c_1(\Sigma)$. The form ω_Σ (pulled back to X) is a pointwise basis for the purely horizontal $(1,1)$-forms. This means that, with respect to the vertical-horizontal decomposition,

$$\omega_0 = \omega_\sigma \oplus \theta\omega_\Sigma$$

for some function $\theta : X \to \mathbb{R}$, where ω_σ is the hyperbolic Kähler form on the fibre S_σ over σ.

For $r > -\inf\theta$, the closed real $(1,1)$-form

$$\omega_r = \omega_0 + r\pi^*\omega_\Sigma$$

is positive, and hence Kähler, with $[\omega_r] = \kappa_r$. Its restriction to S_σ is ω_σ as required. $\square$

The next lemma shows that the metrics ω_r constructed above have approximately constant scalar curvature. First, however, some notation.

Definition 4.2. The *vertical Laplacian*, denoted Δ_V, is defined by

$$(\Delta_V\phi)\,\omega_\sigma = i(\bar\partial\partial\phi)_{VV},$$

where $(\alpha)_{VV}$ denotes the purely vertical component of a $(1,1)$-form α. The fibre wise restriction of Δ_V is the Laplacian determined by ω_σ.

The *horizontal Laplacian*, denoted Δ_H, is defined by

$$(\Delta_H\phi)\,\omega_\Sigma = (i\bar\partial\partial\phi)_{HH},$$

where $(\alpha)_{HH}$ denotes the purely horizontal component of a $(1,1)$-form α. On functions pulled up from the base, Δ_H is the Laplacian determined by ω_Σ.

Lemma 4.3. *The scalar curvature of ω_r satisfies*

$$\mathrm{Scal}(\omega_r) = -1 + r^{-1}\left(\mathrm{Scal}(\omega_\Sigma) - \theta + \Delta_V\theta\right) + O(r^{-2}). \tag{4.1}$$

Proof. The short exact sequence of holomorphic bundles

$$0 \to V \to TX \to H \to 0$$

induces an isomorphism $K_X \cong V^* \otimes H^*$. This means that the Ricci form of ω_r is given by $\rho_r = i(F_V + F_H)$ where F_V and F_H are the curvature forms of V and H respectively.

The metric on the horizontal tangent bundle is $(r+\theta)\omega_\Sigma$. Its curvature is given by

$$iF_H = \rho_\Sigma + i\bar{\partial}\partial \log(1 + r^{-1}\theta),$$

where ρ_Σ is the Ricci form of ω_Σ. The curvature of the vertical tangent bundle has already been considered in the definition $\omega_0 = -iF_V$. Hence

$$\rho_r = -\omega_\sigma - \theta\omega_\Sigma + \rho_\Sigma + i\bar{\partial}\partial \log(1 + r^{-1}\theta). \tag{4.2}$$

Taking the trace gives

$$\mathrm{Scal}(\omega_r) = -1 + \frac{\mathrm{Scal}(\omega_\Sigma) - \theta}{r + \theta} + \Delta_r \log(1 + r^{-1}\theta).$$

where Δ_r is the Laplacian determined by ω_r. Using the formula

$$\Delta_r = \Delta_V + \frac{\Delta_H}{r + \theta} \tag{4.3}$$

and expanding out in powers of r^{-1} proves the result. $\square$

Since $\mathrm{Scal}(\omega_r) = -1 + O(r^{-1})$, setting $\omega_{r,0} = \omega_r$ gives the first family of approximate solutions.

4.2 The second order approximate solution

As will be explained in section 5.2, the scalar curvature of ω_r is not sufficiently close to being constant for the implicit function theorem to be of direct use. This section constructs the necessary higher order approximations.

Let L_r denote the linearisation of the scalar curvature map on Kähler potentials determined by ω_r. The r dependence of L_r will be of central importance in the proof of Theorem 1.1. Its study will essentially occupy the remainder of this thesis. A first step in this direction is provided by the following lemma. Recall that the expression $O(r^{-1})$ is meant purely formally here. (See the discussion at the start of this Chapter for an elaboration of this.)

Lemma 4.4.

$$L_r = \Delta_V^2 + \Delta_V + O(r^{-1}).$$

Proof. Recall the formula (3.2) for L_r:

$$L_r(\phi) = \Delta_r^2 \phi - \mathrm{Scal}(\omega_r)\Delta_r \phi + \frac{2i\bar{\partial}\partial\phi \wedge \rho_r}{\omega_r^2}.$$

Equations (4.1), (4.2) and (4.3) give the r dependence of $\mathrm{Scal}(\omega_r)$, ρ_r and Δ_r respectively. Direct calculation gives the result. $\square$

Remark. Notice that the $O(1)$ term of L_r is the first order variation in the scalar curvature of the fibres (see Example 3.4). This can be seen as an example of the dominance of the local geometry of the fibre in an adiabatic limit.

Rather than use a calculation as above, this result can be seen directly from formula (4.1). The $O(1)$ term in $\mathrm{Scal}(\omega_r)$ is $\mathrm{Scal}(\omega_\sigma)$. Rather than considering a Kähler potential as a change in ω_r, it can be thought of as a change in ω_0. This gives a corresponding change in ω_σ and the $O(1)$ effect on $\mathrm{Scal}(\omega_r)$ is precisely that claimed.

Given a function $\phi \in C^\infty(X)$, taking the fibrewise mean value gives a function $\pi_\Sigma \phi \in C^\infty(\Sigma)$:

$$(\pi_\Sigma \phi)(\sigma) = \frac{1}{\mathrm{vol}(S_\sigma)} \int_{S_\sigma} \phi\, \omega_\sigma,$$

where $S_\sigma = \pi^{-1}(\sigma)$ is the fibre over σ. Define $\pi_0 = 1 - \pi_\Sigma$. The projection maps π_0 and π_Σ determine a splitting

$$C^\infty(X) = C_0^\infty(X) \oplus C^\infty(\Sigma),$$

where $C_0^\infty(X)$ denotes functions with fibrewise mean value zero.

The previous lemma implies that, at least to $O(r^{-1})$, functions in the image of L_r have fibrewise mean value zero. It is because of this that the $C_0^\infty(X)$ and $C^\infty(\Sigma)$ components of the errors in $\mathrm{Scal}(\omega_r)$ must be dealt with differently.

Recall (from Lemma 4.3) that the $O(r^{-1})$ term in $\mathrm{Scal}(\omega_r)$ is

$$\mathrm{Scal}(\omega_\Sigma) - \theta + \Delta_V \theta.$$

Taking the fibrewise mean value shows that the $C^\infty(\Sigma)$ component of this is

$$\mathrm{Scal}(\omega_\Sigma) - \pi_\Sigma \theta.$$

As will be shown, there is a choice of ω_Σ for which this is constant. (The definition of ω_r has, so far, involved an arbitrary metric on Σ.) The remainder of the $O(r^{-1})$ error will be corrected by adding a Kähler potential $i\bar{\partial}\partial r^{-1}\phi_1$ to ω_r.

4.2.1 The correct choice of ω_Σ

It follows from the definition of θ (as the horizontal part of $\omega_0 = -iF_V$ divided by ω_Σ) that

$$\pi_\Sigma \theta = -A^{-1} \Lambda_\Sigma \pi_* (F_V^2),$$

where A is the area of a fibre and Λ_Σ is the trace on $(1,1)$-forms on Σ determined by ω_Σ. The following theorem will be used to find ω_Σ.

Theorem 4.5. *Let Σ be a compact curve with genus at least 2, and $\alpha \in \Omega^2(\Sigma)$ a form with nonnegative integral. Each conformal class on Σ contains a unique representative with*

$$\mathrm{Scal} - \Lambda\alpha = -1.$$

Proof. Pick a metric ω on Σ. Write any other metric in the same conformal class as $\omega' = e^h \omega$. As in equation (2.3), the curvatures of ω and ω' are related by

$$\mathrm{Scal}' = e^{-h} \left(\mathrm{Scal} + \Delta h \right),$$

where Δ is the ω-Laplacian. The traces of α with respect to ω and ω' are related by

$$\Lambda' \alpha = e^{-h} \Lambda \alpha.$$

Hence the theorem will be proved if there exists a unique solution h to the equation

$$\Delta h + e^h = \Lambda\alpha - \mathrm{Scal}\,.$$

This is precisely the partial differential equation solved in Theorem 2.3. Since $\int(\Lambda\alpha - \mathrm{Scal})\omega > 0$, the result follows. $\square$

In order to apply this result, first notice that the cases of Theorem 1.1 with base genus 0 or 1 were dealt with in Chapter 1. Attention in these later chapters is restricted solely to bases with high genus (*i.e.* at least 2).

It remains to prove that the form $\alpha = -\pi_*(F_V^2)$ has positive integral over Σ. Recall the discussion of the signature of X in Section 1.2.2 and in particular the proof of Theorem 1.4. This showed that $[\alpha]$ was the pull back of an ample class from the moduli space of curves (Mumford's first tautological class to be precise). It is because of this that $\int \alpha \geq 0$.

In fact, although this will not be used here, α is the pull-back of the Weil-Petersson form from the moduli space of curves. This follows from results of Wolpert [Wol86].

The upshot is that there is a choice of metric ω_Σ for which the $O(r^{-1})$ term in $\mathrm{Scal}(\omega_r)$ has fibrewise mean value -1. From now on this choice of metric is assumed to be included in the definition of ω_r.

4.2.2 The correct choice of Kähler potential ϕ_1

Let Θ_1 denote the $C_0^\infty(X)$ component of the $O(r^{-1})$ term in $\mathrm{Scal}(\omega_r)$. This means that

$$\mathrm{Scal}(\omega_r) = -1 + r^{-1}(\Theta_1 - 1) + O(r^{-2}).$$

It follows from Lemma 4.4 that

$$\mathrm{Scal}(\omega_r + i\bar{\partial}\partial r^{-1}\phi) = \mathrm{Scal}(\omega_r) + r^{-1}(\Delta_V^2 + \Delta_V)\phi + O(r^{-2}). \tag{4.4}$$

Lemma 4.6. *Let $\Theta \in C_0^\infty(X)$. There exists a unique $\phi \in C_0^\infty(X)$ such that*

$$\left(\Delta_V^2 + \Delta_V\right)\phi = \Theta.$$

Proof. Given a function $\phi \in C^\infty(X)$ let ϕ_σ denote the restriction of ϕ to S_σ. The fibrewise restriction of the operator $\Delta_V^2 + \Delta_V$ is the first order variation of the scalar curvature of the fibre. Denote the restriction of this operator to the fibre S_σ by L_σ. Recall from Example 3.4 that

$$L_\sigma : L^2_{k+4}(S_\sigma) \to L^2_k(S_\sigma)$$

is an isomorphism when considered as a map between spaces of functions with mean value zero (over S_σ).

Applying this fibrewise certainly gives a unique function ϕ on X such that ϕ has fibrewise mean value zero, for each σ, $\phi_\sigma \in C^\infty(S_\sigma)$ and $L_\sigma \phi_\sigma = \Theta_\sigma$, *i.e.* $(\Delta_V^2 + \Delta_V)\phi = \Theta$. It only remains to check that ϕ is smooth transverse to the fibres. (The operator $\Delta_V^2 + \Delta_V$ is only elliptic in the fibre directions, so regularity only follows automatically in those directions.)

In fact, this is straight forward. Since $\phi_\sigma = L_\sigma^{-1}\Theta_\sigma$ the required differentiability follows from that of Θ and the fact that L_σ is a smooth family of differential operators. □

Remark. This proof is not meant to suggest that a smooth family of fibrewise elliptic operators enjoys the same regularity properties as a genuinely elliptic operator. If, for example, in the above Lemma, $\Theta \in L_k^2$, then $\phi_\sigma \in L_{k+4}^2(S_\sigma)$, but as a function over X, $\phi \in L_k^2$.

Applying this lemma to $\Theta = -\Theta_1$ and using equation (4.4) shows that there exists a unique $\phi_1 \in C_0^\infty(X)$ such that the metric

$$\omega_{r,1} = \omega_r + i\bar{\partial}\partial r^{-1}\phi_1$$

is an $O(r^{-2})$ approximate solution to the constant scalar curvature equation:

$$\mathrm{Scal}(\omega_{r,1}) = -1 - r^{-1} + O(r^{-2}).$$

4.3 The third order approximate solution

Now that the correct metric has been found on the base, the higher order approximate solutions are constructed recursively. In order to demonstrate the key points clearly, however, this section does the first step in detail.

The strategy is straightforward, even if the notation sometimes isn't! The first step is to find a Kähler potential f_1 on the base to deal with the $C^\infty(\Sigma)$ component of the $O(r^{-2})$ error. That is, so that

$$\mathrm{Scal}(\omega_{r,1} + i\bar{\partial}\partial f_1) = -1 - r^{-1} + (c + \Theta_2')r^{-2} + O(r^{-3}),$$

for some constant c, where Θ_2' has fibrewise mean value zero.

The fact that Kähler potentials on the base affect the scalar curvature at $O(r^{-2})$ and no higher order, at least in the case of ω_r, is seen as follows. The potential can be thought of as altering the metric on the base. Since the base metric is scaled by r in the definition of ω_r, adding the potential f to ω_r is equivalent to adding the potential $r^{-1}f$ to ω_Σ. Equation (4.1) shows that the lowest order effect of ω_Σ on the scalar curvature of ω_r occurs at $O(r^{-1})$. Hence the combined effect is $O(r^{-2})$.

The second step is to find a Kähler potential ϕ_2 to deal with the remaining $O(r^{-2})$ error Θ_2'. That is, so that

$$\mathrm{Scal}\left(\omega_{r,1} + i\bar{\partial}\partial(f_1 + r^{-2}\phi_2)\right) = -1 - r^{-1} + cr^{-2} + O(r^{-3}).$$

The potential ϕ_2 is found in much the same way as ϕ_1 was found above, via Lemma 4.6.

Both of the potentials f_1 and ϕ_2 are found as solutions to linear partial differential equations. To find the relevant equations, it is important to understand the linearisation of the scalar curvature map on Kähler potentials determined by $\omega_{r,1}$ (and the operators determined by the later, higher order, approximate solutions). To this end, the first lemma in this section deals with the r dependence of such an operator when the fibrewise metrics are not necessarily the canonical constant curvature ones. First, some notation.

Notation for Lemma 4.11

Let Ω_0 be any closed real $(1,1)$-form whose fibrewise restriction is Kähler. Let Ω_σ be the Kähler form on S_σ induced by Ω_0. Let Ω_Σ be any choice of metric on the base. As in Lemma 4.1, for large enough r, the form $\Omega_r = \Omega_0 + r\Omega_\Sigma$ is Kähler. The vertical-horizontal decomposition of the tangent bundle determined by Ω_r depends only on Ω_0.

Definition 4.7. The form Ω_Σ is a pointwise basis for the horizontal $(1,1)$-forms. Define a function ξ as follows. Write the horizontal-vertical decomposition of Ω_0 (with respect to Ω_r) as

$$\Omega_0 = \Omega_\sigma \oplus \xi\Omega_\Sigma.$$

Definition 4.8. The family of fibrewise Kähler metrics Ω_σ determines a Hermitian structure in the vertical tangent bundle. Denote the curvature of this bundle as F_V. Define a function η as follows. Write the horizontal-vertical decomposition of iF_V (with respect to Ω_r) as

$$iF_V = \rho_\sigma \oplus \eta\Omega_\Sigma.$$

Remark. Since the fibrewise metrics are not the canonical constant curvature ones, this curvature form is not the same as that appearing earlier. If, instead of any old Ω_0 and Ω_Σ, the forms ω_0 and ω_Σ from earlier are used in both of these definitions, then $\xi = -\eta = \theta$.

It is also necessary to define a certain operator on functions on the base.

Definition 4.9. Taking the fibrewise mean value of η gives a function $\pi_\Sigma \eta$ on the base. Using this, define a fourth order differential operator

$$D_\Sigma : C^\infty(\Sigma) \to C^\infty(\Sigma),$$

$$D_\Sigma(f) = \Delta_\Sigma^2 f - (\mathrm{Scal}(\Omega_\Sigma) + \pi_\Sigma \eta)\Delta_\Sigma f,$$

where Δ_Σ is the Ω_Σ-Laplacian.

Remark 4.10. The operator D_Σ is the linearisation of a nonlinear map on functions, which is now described. Let$(iF_V)_{HH}$ denote the purely horizontal component of iF_V with respect to the vertical-horizontal decomposition determined by Ω_0. Notice that this does not depend on the choice of Ω_Σ. Taking the fibrewise mean value of $(iF_V)_{HH}$ defines a 2-form on the base Σ which is again independent of the choice of Ω_Σ. The trace of this form with respect to Ω_Σ is precisely the fibrewise mean value of η:

$$\pi_\Sigma \eta = \Lambda_\Sigma \pi_\Sigma (iF_V)_{HH}.$$

This shows exactly how $\pi_\Sigma \eta$ depends on the choice of Ω_Σ, *i.e.* only through Λ_Σ.

Next, consider varying Ω_Σ by a Kähler potential $f \in C^\infty(\Sigma)$. Denote by $\Lambda_{\Sigma,f}$ the trace operator determined by $\Omega_\Sigma + i\bar{\partial}\partial f$. The equation

$$\Lambda_{\Sigma,f} = \frac{\Lambda_\Sigma}{1 + \Delta_\Sigma f}$$

shows that the linearisation at 0 of the map

$$f \mapsto \Lambda_{\Sigma,f} \pi_\Sigma (iF_V)_{HH} = \pi_\Sigma \eta$$

is $-\pi_\Sigma \eta \Delta_\Sigma$. Combining this with the formula for the linearisation of the scalar curvature map on curves derived in Example 3.4, shows that D_Σ is the linearisation, at 0, of the map

$$F \colon f \mapsto \mathrm{Scal}(\Omega_\Sigma + i\bar{\partial}\partial f) + \pi_\Sigma \eta.$$

If, instead of using any old Ω_0, the definition were made using ω_0 from earlier, then the map F is one which has been described before. It is precisely the map which was shown to take the value -1 at ω_Σ (see section 4.2.1). Notice that using ω_0 and ω_Σ to define D_Σ gives $D_\Sigma = \Delta_\Sigma^2 + \Delta_\Sigma$. As in Example 3.4, this operator is an isomorphism on functions of mean value zero (when considered as a map between the relevant Sobolev spaces).

The vertical and horizontal Laplacians are defined just as before, with Ω_0 and Ω_Σ replacing ω_0 and ω_Σ respectively (see Definition 4.2). To indicate that they are defined with respect to different forms (and also a different vertical-horizontal decomposition of the tangent bundle, notice), the vertical and horizontal Laplacians determined by Ω_0 and Ω_Σ are denoted Δ'_V and Δ'_H. The un-primed symbols are reserved for the vertical and horizontal Laplacians determined by ω_0 and ω_Σ.

Let $L(\Omega_r)$ denote the linearisation of the scalar curvature map on Kähler potentials defined by Ω_r. The relevant notation is now in place to state and prove the following lemma.

Lemma 4.11.

$$L(\Omega_r) = (\Delta_V'^2 - \mathrm{Scal}(\Omega_\sigma)\Delta'_V) + r^{-1}D_1 + r^{-2}D_2 + O(r^{-3}),$$

where the operators D_1 and D_2 depend only on Ω_0 and Ω_Σ. Moreover, if f is a function pulled back from Σ,

$$\begin{aligned} D_1(f) &= 0, & (4.5)\\ \pi_\Sigma D_2(f) &= D_\Sigma(f). & (4.6) \end{aligned}$$

Proof. The proof given here is a long calculation. A slightly more conceptual proof is described in a following remark. Recall the formula (3.2) for the linearisation of the scalar curvature map. It involves the Laplacian, the scalar curvature and the Ricci form of Ω_r. Repeating the calculations that were used when the fibres had constant scalar curvature metrics gives formulae for these objects. They are, respectively,

$$\begin{aligned} \Delta_{\Omega_r} &= \Delta'_V + \frac{\Delta'_H}{r+\xi}, & (4.7)\\ \mathrm{Scal}(\Omega_r) &= \mathrm{Scal}(\Omega_\sigma) + \frac{\mathrm{Scal}(\Omega_\Sigma)+\eta}{r+\xi} + \Delta_{\Omega_r}\log(1+r^{-1}\xi), & (4.8)\\ \rho(\Omega_r) &= \rho(\Omega_\sigma) + (\mathrm{Scal}(\Omega_\Sigma)+\eta)\,\Omega_\Sigma + i\bar\partial\partial\log(1+r^{-1}\xi). & (4.9) \end{aligned}$$

The result now follows from routine manipulation and expansion of power series. In particular the following formulae can be verified for D_1 and D_2:

$$\begin{aligned} D_1 &= 2\Delta'_V\Delta'_H - (\Delta'_V\xi)\Delta'_V, \\ D_2 &= \Delta'^2_H - (\mathrm{Scal}(\Omega_\Sigma) + \eta)\Delta'_H - \eta\Delta'_V\Delta'_H \\ &\quad + \frac{1}{2}(\Delta'_V(\xi^2))\Delta'_V + (\Delta'_V\eta)\Delta'_H. \end{aligned}$$

The statements about $D_1(f)$ and $\pi_\Sigma D_2(f)$ for f pulled up from the base follow from these equations. □

Remark. The actual equations for D_1 and D_2 will not be needed in what follows. All that will be used is their stated behaviour on functions on the base as stated in Lemma 4.11. This behaviour can be understood, without laborious calculation, as follows.

The fact that potentials on the base affect the scalar curvature of Ω_r at $O(r^{-2})$ and no higher order is discussed at the start of this section. So for potentials f pulled up from the base $D_1(f) = 0$. The fibrewise mean value of the $O(r^{-1})$ term in $\mathrm{Scal}(\Omega_r)$ is

$$\mathrm{Scal}(\Omega_\Sigma) + \pi_\Sigma\eta.$$

So, after taking the fibrewise mean value, a change of $r^{-1}f$ in Ω_Σ gives a change in $\pi_\Sigma \mathrm{Scal}(\Omega_r)$ whose $O(r^{-2})$ term is given by the derivative of the above expression with respect to Kähler potentials on the base. Hence $\pi_\Sigma D_2(f) = D_\Sigma(f)$.

4.3.1 The correct choice of Kähler potential f_1

Denote the $C^\infty(\Sigma)$ component of the $O(r^{-2})$ term of $\mathrm{Scal}(\omega_{r,1})$ by Θ_2:

$$\pi_\Sigma \mathrm{Scal}(\omega_{r,1}) = -1 - r^{-1} + r^{-2}\Theta_2 + O(r^{-3}).$$

The first step in constructing the $O(r^{-3})$ approximate solution is to find a potential $f_1 \in C^\infty(\Sigma)$ which compensates for Θ_2, *i.e.* with

$$\pi_\Sigma \mathrm{Scal}(\omega_{r,1} + i\bar\partial\partial f_1) = -1 - r^{-1} + cr^{-2} + O(r^{-3}),$$

where c is the mean value of Θ_2 over Σ (with respect to ω_Σ).

Let $L_{r,1}$ be the linearisation of the scalar curvature map on Kähler potentials determined by $\omega_{r,1}$. The next result uses Lemma 4.11 to describe the $O(r^{-2})$ behaviour of $L_{r,1}$.

Lemma 4.12. *Let $f \in C^\infty(\Sigma)$. Then*

$$\pi_\Sigma L_{r,1}(f) = r^{-2}(\Delta_\Sigma^2 + \Delta_\Sigma)f + O(r^{-3}).$$

Proof. Begin by applying Lemma 4.11 with

$$\begin{aligned} \Omega_0 &= \omega_0 + i\bar{\partial}\partial r^{-1}\phi_1, \\ \Omega_\Sigma &= \omega_\Sigma. \end{aligned}$$

There is a slight difficultly in interpreting the expansion given in Lemma 4.11. The r-dependence of Ω_0 means that some of the coefficients in the $O(r^{-3})$ piece of that expansion will be r-dependent, *a priori* making them of higher order overall.

In fact, this can't happen. The reason is that all such coefficients come ultimately from analytic expressions in the fibrewise metrics induced by Ω_0 (as is shown, for example, by the calculation described in the proof of Lemma 4.11). These metrics have the form

$$\Omega_\sigma = (1 + r^{-1}\Delta_V\phi_1)\omega_\sigma.$$

(Here Δ_V is the vertical Laplacian determined by ω_0.) Since the fibrewise metric is algebraic in r^{-1}, the coefficients in the expression form Lemma 4.11 are analytic in r^{-1}, *i.e.* they have expansions involving only nonpositive powers of r.

This means that the $O(r^{-2})$ term can simply be read off from the formula given in Lemma 4.11. This gives

$$\pi_\Sigma L_{r,1}(f) = r^{-2}D_\Sigma(f) + O(r^{-3}).$$

As is pointed out in Remark 4.10, for the choice of ω_Σ that was determined whilst finding the $O(r^{-2})$ approximate solution, $D_\Sigma = \Delta_\Sigma^2 + \Delta_\Sigma$ as required. $\square$

As Example 3.4 explains, the equation

$$(\Delta_\Sigma^2 + \Delta_\Sigma)f_1 = c - \Theta_2$$

for f_1 has a solution (and a unique one if f_1 is also required to have mean value zero over Σ). Elliptic regularity ensures that f_1 is smooth, completing the first step in finding the $O(r^{-3})$ approximate solution.

This leaves

$$\mathrm{Scal}(\omega_{r,1} + i\bar{\partial}\partial f_1) = -1 - r^{-1} + r^{-2}(c + \Theta_2') + O(r^{-3}),$$

where Θ_2' has fibrewise mean value zero.

4.3.2 The correct choice of Kähler potential ϕ_2

The next step in constructing the $O(r^{-3})$ approximate solution is to find a potential $\phi_2 \in C_0^\infty(X)$ which compensates for Θ_2', *i.e.* with

$$\mathrm{Scal}\left(\omega_{r,1} + i\bar{\partial}\partial(f_1 + r^{-2}\phi_2)\right) = -1 - r^{-1} + cr^{-2} + O(r^{-3}).$$

Let $L_{r,1}'$ be the linearisation of the scalar curvature map on Kähler potentials determined by the metric $\omega_{r,1} + i\bar{\partial}\partial f_1$. The next result uses Lemma 4.11 to describe the $O(1)$ behaviour of $L_{r,1}'$.

Lemma 4.13.

$$L_{r,1}' = \Delta_V^2 + \Delta_V + O(r^{-1}).$$

Remark. Again, the symbol Δ_V means the vertical Laplacian determined by the form ω_0. This lemma merely says that the $O(1)$ behaviour of $L_{r,1}'$ is the same as that of L_r (see Lemma 4.4).

Proof. Apply Lemma 4.11 with

$$\begin{aligned} \Omega_0 &= \omega_0 + i\bar{\partial}\partial r^{-1}\phi_1, \\ \Omega_\Sigma &= \omega_\Sigma + i\bar{\partial}\partial r^{-1} f_1. \end{aligned}$$

As in the proof of Lemma 4.12, there is a problem with interpreting the expansion in Lemma 4.11, namely that the r-dependence of Ω_0 and Ω_Σ means that the coefficients in the expansion are also r-dependent. As in the proof of Lemma 4.12, however, this actually causes no difficulty. Both forms are algebraic in r^{-1}, hence the coefficients in the expansion are analytic in r^{-1}. Hence the r-dependence of the coefficient of r^{-n} causes changes only at $O(r^{-n-k})$ for $k \geq 0$. This means that the genuine $O(1)$ behaviour of $L_{r,1}'$ is the same as the $O(1)$ behaviour of

$$\Delta_V'^2 - \mathrm{Scal}(\Omega_\sigma)\Delta_V'.$$

Here, Ω_σ is the metric on S_σ determined by Ω_0, *i.e.*

$$\Omega_\sigma = (1 + r^{-1}\Delta_V\phi_1)\omega_\sigma.$$

Since, to $O(1)$, Ω_σ and ω_σ agree,

$$\begin{aligned}\mathrm{Scal}(\Omega_\sigma) &= \mathrm{Scal}(\omega_\sigma) + O(r^{-1}),\\ \Delta'_V &= \Delta_V + O(r^{-1}).\end{aligned}$$

Hence

$$\Delta'^2_V - \mathrm{Scal}(\Omega_\sigma)\Delta'_V = \Delta^2_V + \Delta_V + O(r^{-1}).$$

This completes the proof. ∎

Lemma 4.6 implies that there exists a unique $\phi_2 \in C_0^\infty(X)$ such that

$$(\Delta^2_V + \Delta_V)\phi_2 = -\Theta'_2.$$

Let $\omega_{r,2} = \omega_{r,1} + i\bar{\partial}\partial(f_1 + r^{-2}\phi_2)$. Then

$$\mathrm{Scal}(\omega_{r,2}) = -1 - r^{-1} + cr^{-2} + O(r^{-3}).$$

4.4 The higher order approximate solutions

This section completes the construction of the higher order approximate solutions, proving:

Theorem 4.14 (Approximately constant scalar curvature metrics). *Let n be a positive integer. There exist functions $f_1, \dots, f_{n-1} \in C^\infty(\Sigma)$ and $\phi_1, \dots, \phi_n \in C_0^\infty(X)$ such that the metric*

$$\omega_{r,n} = \omega_r + i\bar{\partial}\partial \sum_{i=1}^{n-1} r^{-i+1} f_i + i\bar{\partial}\partial \sum_{i=1}^{n} r^{-i}\phi_i$$

satisfies

$$\mathrm{Scal}(\omega_{r,n}) = -1 + \sum_{i=1}^{n} c_i r^{-i} + O\left(r^{-n-1}\right),$$

for constants c_i.

Proof. The strategy for the recursive construction of the higher order approximations is, hopefully, now clear. As an inductive hypothesis, assume that an $O(r^{-n})$ approximate solution has been found of the form

$$\omega_{r,n-1} = \omega_r + i\bar{\partial}\partial \sum_{i=1}^{n-2} r^{-i+1} f_i + i\bar{\partial}\partial \sum_{i=1}^{n-1} r^{-i}\phi_i,$$

where $\phi_i \in C_0^\infty(X)$ and $f_i \in C^\infty(\Sigma)$. Assume its scalar curvature satisfies

$$\mathrm{Scal}(\omega_{r,n-1}) = -1 + \sum_{i=1}^{n-1} c_i r^{-i} + O\left(r^{-n}\right),$$

for constants c_i.

The correct choice of Kähler potential f_{n-1}

Denote by $L_{r,n-1}$ the linearisation of the scalar curvature map on Kähler potentials determined by $\omega_{r,n-1}$. Just as in the proof of Lemma 4.12, it can be shown, using Lemma 4.11, that for a function f pulled up from the base

$$\pi_\Sigma L_{r,n-1}(f) = r^{-2}\left(\Delta_\Sigma^2 + \Delta_\Sigma\right)(f) + O\left(r^{-3}\right).$$

This, and Example 3.4, show that there exists f_{n-1} such that

$$\mathrm{Scal}\left(\omega_{r,n-1} + i\bar{\partial}\partial r^{-(n-2)} f_{n-1}\right) = -1 + \sum_{i=1}^{n-1} c_i r^{-i} + r^{-n}(c_n + \Theta_n) + O\left(r^{-n-1}\right),$$

where Θ_n has fibrewise mean value zero.

The correct choice of Kähler potential ϕ_n

Denote by $L'_{r,n-1}$ the linearisation of the scalar curvature map on Kähler potentials determined by $\omega_{r,n-1} + i\bar{\partial}\partial r^{-(n-2)} f_{n-1}$. Just as in the proof of Lemma 4.13, it can be shown, using Lemma 4.11, that

$$L'_{r,n-1} = \Delta_V^2 + \Delta_V + O\left(r^{-1}\right).$$

This, and Lemma 4.6, show that there exists ϕ_n such that

$$\mathrm{Scal}\left(\omega_{r,n-1} + i\bar{\partial}\partial\left(r^{-(n-2)} f_{n-1} + r^{-n}\phi_n\right)\right) = -1 + \sum_{i=1}^{n} c_i r^{-i} + O\left(r^{-n-1}\right).$$

$\square$

In fact, it is straight forward to show that the functions f_i, ϕ_i are unique subject to the constraints

$$\int_\Sigma f_i\, \omega_\Sigma = 0 \qquad \int_{S_\sigma} \phi_i\, \omega_\sigma = 0.$$

This uses the injectivity of the operator discussed in Example 3.4 (acting on functions with mean value zero).

It is also possible to calculate the exact values of the constants c_i via a calculation of the mean value of $\mathrm{Scal}(\omega_r)$. Let

$$\mathrm{vol}_r = \frac{1}{2}\int \omega_r^2 = rA + B$$

where $A = [\omega_0].[\omega_\Sigma]$ and $B = 2^{-1}[\omega_0]^2$, and let

$$\frac{1}{2}\int \mathrm{Scal}(\omega_r)\omega_r^2 = 2\pi c_1(X).[\omega_r] = rC + D$$

where $C = 2\pi c_1(X).[\omega_0]$ and $D = 2\pi c_1(X).[\omega_\Sigma]$. Then the mean value of the scalar curvature is

$$\frac{rC + D}{rA + B} = -1 + \sum c_i r^{-i},$$

where $c_i = (-1)^i A^{-i} B^i \left(A^{-1}C - B^{-1}D\right)$.

4.5 Summary

Four essential facts were used in the construction of the approximate solutions in Theorem 4.14:

1. The nonlinear partial differential equation

   $$\mathrm{Scal}(\omega_\sigma) = \text{const.}$$

   in the fibre directions has a solution. This enabled $\omega_{r,0}$ to be constructed.

2. The linearisation of this equation, at a solution, is surjective onto functions with mean value zero. This enabled the $C_0^\infty(X)$ components of error terms to be eliminated.

3. The nonlinear partial differential equation

   $$\mathrm{Scal}(\omega_\Sigma) - \Lambda_\Sigma \alpha = \text{const.}$$

   on the base has a solution. This enabled $\omega_{r,1}$ to be constructed.

4. The linearisation of this equation, at a solution, is surjective onto functions with mean value zero. This enabled the $C^\infty(\Sigma)$ components of error terms to be made constant.

There are related reasons as to why these facts are important. Firstly, the fact that the linear operator in 2 is not genuinely surjective, but maps only onto the functions with mean value zero, is what necessitates an alternative approach to the $C^\infty(\Sigma)$ components of errors. Secondly, the solution mentioned in 1 gives rise to the α term in 3.

The surjectivity of the linear operators in 2 and 4 can be viewed in terms of automorphisms of the solutions in 1 and 3 respectively. This is because both operators are elliptic with index zero. Hence they are surjective if and only if they have no kernel (thought of as maps between spaces of functions with mean value zero). The absence of any kernel is equivalent to there being no nontrivial family of solutions to the equations in 1 and 3.

Finally, it should be noted that the two parameters r and n appearing in the approximate solutions are of a very different nature. In particular, whilst the perturbation to a genuine solution is carried out, n is considered as fixed, whilst r tends to infinity.

5 Setting up the adiabatic limit

The remaining chapters carry out the analysis necessary to perturb the families of approximate solutions $\omega_{r,n}$, constructed in the previous chapter, to genuine solutions. Section 5.1 sets up the problem and discusses using a parameter dependent inverse function theorem to solve it. In particular it describes how the solution hinges on certain analytic estimates.

Section 5.2 considers the estimates over a product of a fixed curve with a flat torus. Although such manifolds obviously admit constant scalar curvature metrics, the analysis is still of interest for two reasons. Firstly, it shows what behaviour to expect in the general case. Secondly, as is proved in Chapter 6, the Kähler structure $(X, J, \omega_{r,n})$ locally approaches such a product as $r \to \infty$. This enables the estimates proved over the product in Section 5.2 to be patched together to give local analytic estimates over X. This is also done in Chapter 6. Chapter 7 explains how to prove uniform global analytic estimates. The proof of Theorem 1.1 is completed in Chapter 8.

5.1 Applying the inverse function theorem

First, some notation. Write $g_{r,n}$ for the metric tensor corresponding to the Kähler form $\omega_{r,n}$. Each such metric defines Sobolev spaces $L^2_k(g_{r,n})$ of functions over X. Since the Sobolev norms determined by $g_{r,n}$, for different values of r, are equivalent, the spaces $L^2_k(g_{r,n})$ contain the same functions. The constants of equivalence, however, will depend on r. Similarly the C^k-norms defined by $g_{r,n}$ can be considered, giving Banach spaces denoted $C^k(g_{r,n})$. When the actual norms themselves are not important, explicit reference to the metrics will be dropped, and the spaces referred to simply as C^k or L^2_k.

Statements like "$a_r \to 0$ in $C^k(g_{r,n})$ as $r \to \infty$," mean "$\|a_r\|_{C^k(g_{r,n})} \to 0$ as $r \to \infty$." Notice that both the norm and the object whose norm is being measured are changing with r. Similarly statements such as "a_r is $O(r^{-1})$ in $L^2_k(g_{r,n})$ as $r \to \infty$," mean "$\|a_r\|_{L^2_k(g_{r,n})}$ is $O(r^{-1})$ as $r \to \infty$."

The scalar curvature map on Kähler potentials determined by the metric $\omega_{r,n}$ defines a map

$$L^2_{k+4} \to L^2_k, \quad \phi \mapsto \mathrm{Scal}(\omega_{r,n} + i\bar{\partial}\partial\phi),$$

when $k \geq 1$. Denote the derivative of this map at 0 by $L_{r,n}$. As with Example 3.4, this derivative will be shown to be an isomorphism when considered modulo the constant functions. In order to apply the inverse function theorem it will be necessary to project onto a complement of the constant functions.

Recall from Remark 3.3 that when the scalar curvature of a Kähler metric ω is constant, the corresponding linear operator maps into functions with ω-mean value zero. In the situation considered here, $\omega_{r,n}$ has nearly constant scalar curvature. So it makes sense to try and show that $L_{r,n}$ gives an isomorphism after composing with the projection p onto functions with $\omega_{r,n}$-mean value zero.

Let $L^2_{k,0}$ denote functions in L^2_k with $\omega_{r,n}$-mean value zero. Composing the scalar curvature map with the projection p gives, for $k > 1$, a map

$$S_{r,n} \colon L^2_{k+4,0} \to L^2_{k,0},$$

$$S_{r,n}(\phi) = p\,\mathrm{Scal}(\omega_{r,n} + i\bar{\partial}\partial\phi).$$

To complete the proof of Theorem 1.1 it will be shown that for each k and sufficiently large r there is a unique $\phi \in L^2_{k+4,0}$ with $S_{r,n}(\phi) = 0$.

To see that this will indeed finish the proof, notice that since $\mathrm{Scal}(\omega_{r,n} + i\bar{\partial}\partial\phi)$ differs from $S_{r,n}(\phi)$ by a constant, $\omega_{r,n} + i\bar{\partial}\partial\phi$ is a constant scalar curvature metric in the class κ_r. (Smoothness of ϕ will follow from the ellipticity of the scalar curvature map; see Lemma 8.3.)

It is possible to use L^p_k-norms in the analysis, rather than just L^2_k. This only requires the proof of one additional lemma. Since this is not necessary in the proof of Theorem 1.1, however, attention is restricted to the L^2_k-norms throughout.

5.1.1 Approximate solutions in $L^2_k(g_{r,n})$

By construction of the metrics $\omega_{r,n}$,

$$\mathrm{Scal}(\omega_{r,n}) = O\left(r^{-n-1}\right).$$

However, at this stage this is meant only in the formal sense described in Chapter 4. In order to use the inverse function theorem, it is first necessary to know that $S_{r,n}(0)$ converges to 0 in $L^2_k(g_{r,n})$ as $r \to \infty$.

The expansions in negative powers of r in Chapter 4 all arise via absolutely convergent power series and algebraic manipulation. This means that with respect to a *fixed* metric g,

$$\mathrm{Scal}(\omega_{r,n}) = O\left(r^{-n-1}\right) \quad \text{in } C^k(g) \text{ as } r \to \infty.$$

For example, $\log(1+r^{-1}\theta)$ is $O(r^{-1})$ in $C^k(g)$ because

$$\begin{aligned}
\|\log(1+r^{-1}\theta)\|_{C^k} &\leq \sum_{j\geq 1} r^{-(j+1)} C^j \frac{\|\theta\|^j_{C^k}}{j}, \\
&= \log\left(1 + Cr^{-1}\|\theta\|_{C^k}\right).
\end{aligned}$$

where C a constant such that $\|\phi\psi\|_{C^k} \leq C\|\phi\|_{C^k}\|\psi\|_{C^k}$.

The same is true with respect to the $C^k(g_{r,n})$-norm *provided that the $C^k(g_{r,n})$-norm of a fixed function is bounded as $r \to \infty$.* (Notice that the constant C above does not depend on g.) In fact, this is the case. Lengths of vertical covectors do not change with r, whilst horizontal covectors are $O(r^{-1/2})$ as $r \to \infty$. If X were a product, and g_r the product metric scaled by r in the base directions, the Levi-Civita connection would be independent of r, the horizontal distribution would be parallel, and this would imply the necessary boundedness immediately.

The difficulty in applying this argument in the general case is that the Levi-Civita connection does depend on r and the horizontal distribution is not parallel. It can be shown that the horizontal distribution converges, in a certain sense, to a parallel distribution, and the boundedness of $\|\phi\|_{C^k(g_{r,n})}$, for any ϕ, follows from this. Since this is a consequence of Theorem 6.1, which is proved later, the result is stated here, but its proof is deferred (see Section 6.2). The result also includes a description of the $L^2_k(g_{r,n})$-behaviour of $\mathrm{Scal}(\omega_{r,n})$ for large r.

Lemma 5.1.

$$\mathrm{Scal}(\omega_{r,n}) = O\left(r^{-n-1}\right) \quad \textit{in } C^k(g_{r,n}) \textit{ as } r \to \infty$$

$$\mathrm{Scal}(\omega_{r,n}) = O\left(r^{-n-1/2}\right) \quad \textit{in } L^2_k(g_{r,n}) \textit{ as } r \to \infty$$

The inverse function theorem will be applied to the map

$$S_{r,n} \colon \phi \mapsto p\,\mathrm{Scal}(\omega_{r,n} + i\bar{\partial}\partial\phi).$$

Hence it is also necessary to know that projection p is uniformly bounded with respect to the $L^2_k(g_{r,n})$-norms. This follows from the fact that p is an $L^2_k(g_{r,n})$-orthogonal projection, and hence has operator norm 1. The next result follows immediately from the last.

Lemma 5.2.

$$S_{r,n}(0) = O\left(r^{-n-1/2}\right) \quad \textit{in } L^2_k(g_{r,n}) \textit{ as } r \to \infty.$$

Since $S_{r,n}(0) \to 0$ in $L^2_k(g_{r,n})$, it might be hoped that the inverse function theorem guarantees the existence of a function ϕ such that $S_{r,n}(\phi) = 0$. This is indeed the case, although the parameter dependence of the problem means this is not as straight forward as it might at first appear. An examination of the proof of the inverse function theorem will help clarify the issues.

Theorem 5.3 (Quantitative inverse function theorem).

- *Let $F : B_1 \to B_2$ be a differentiable map of Banach spaces, whose derivative at* 0, *DF, is an isomorphism of Banach spaces, with inverse P.*

- *Let δ' be the radius of the closed ball in B_1, centred at* 0, *on which $F - DF$ is Lipschitz, with constant* $1/(2\|P\|)$.

- *Let $\delta = \delta'/(2\|P\|)$.*

Then whenever $y \in B_2$ satisfies $\|y - F(0)\| < \delta$, there exists $x \in B_1$ with $F(x) = y$. Moreover such an x is unique subject to the constraint $\|x\| < \delta'$.

Proof. Define a sequence $x_n \in B_1$ recursively via the Newton-Raphson method:

$$\begin{aligned} x_0 &= 0 \\ x_{n+1} &= x_n + P\left(y - F(x_n)\right) \end{aligned}$$

If the map $G : x \mapsto x + P(y - F(x_n))$ is a contraction on some closed ball K about the origin, this iteration will converge to a solution x, unique in K, of the equation $F(y) = x$.

Since

$$\|G(x) - G(x')\| \leq \|P\|\|(F - DF)(x) - (F - DF)(x')\|$$

G is seemingly a contraction on the ball K about $0 \in B_1$ on which $F - DF$ is Lipschitz with constant $1/(2\|P\|)$. It may not map K to itself, however.

Since, for $x \in K$,

$$\begin{aligned} \|G(x)\| &\leq \|P\|\|(F - DF)(x) - (F - DF)(0)\| + \|P\|\|y - F(0)\|, \\ &\leq \frac{1}{2}\|x\| + \|P\|\|y - F(0)\|, \end{aligned}$$

the condition $\|y - F(0)\| < \delta'/(2\|P\|)$ implies that $G : K \to K$ is a genuine contraction. $\square$

Assume for now that the derivative of the map

$$S_{r,n} : L^2_{k+4,0}(g_{r,n}) \to L^2_{k,0}(g_{r,n})$$

is an isomorphism with inverse denoted $P_{r,n}$ (this is proved in Chapter 7). Notice that both the map and the norms used depend on r. The inverse function theorem can be applied. It guarantees the existence of a $\delta_{r,n}$ such that if

$$\|S_{r,n}(0)\|_{L^2_k(g_{r,n})} < \delta_{r,n},$$

then there exists a ϕ with $S_{r,n}(\phi) = 0$.

The problem is that even though $S_{r,n}(0)$ tends to zero in $L^2_k(g_{r,n})$, $\delta_{r,n}$ may converge to zero even quicker. It will be shown later that $\delta_{r,n} \geq Cr^{-6}$. This means that, for $n \geq 6$, Lemma 5.2 implies that $S_{r,n}(0)$ converges to zero quickly enough for the inverse function theorem to be of use.

5.2 A motivating example

The key step in controlling $\delta_{r,n}$ is to bound the behaviour of $\|P_{r,n}\|$. The uniform control of the non-linear pieces proves much easier. This section discusses this operator over the Kähler product of a fixed high genus curve with a flat torus.

Whilst such manifolds obviously admit constant scalar curvature Kähler metrics, the aim is understand the estimates required to apply the inverse function theorem in the general case. The main conclusion is that $\|P_{r,n}\| \leq Cr^2$. Going back to the statement of the inverse function theorem given above, $\|P_{r,n}\|^{-1}$ appears twice in the definition of $\delta_{r,n}$. This suggests that $\delta_{r,n} \geq Cr^{-4}$, a fact that should still hold in the general case. A weaker estimate $\delta \geq Cr^{-6}$ is proved for the general case. This is sufficient so far as the proof of Theorem 1.1 is concerned.

As mentioned above, the product will serve as a local model for the general case, meaning uniform estimates proved over the product can be patched to give uniform estimates over X. To this end, this section will also prove that various Sobolev inequalities hold uniformly over the product.

5.2.1 The adiabatic limit for a product

Let $S \times T^2$ be the product of a fixed high genus curve S with a flat torus $T^2 = S^1 \times S^1$. Let ω_S denote the canonical hyperbolic metric on S and ω_{T^2} a flat metric on T^2. Define $\omega'_r = \omega_S \oplus r\omega_{T^2}$, which is a Kähler metric on $S \times T^2$. Write g'_r for the corresponding metric tensor. Primes are used to distinguish this Kähler structure from the general case considered in the previous chapter. (This is important in the next chapter, when both structures are considered simultaneously.)

The scalar curvature map on Kähler potentials determines a map

$$S'_r : L^2_{k+4} \to L^2_k.$$

From Examples 3.5 and 3.4, its derivative is given by the formula

$$L'_r = \Delta_S^2 + \Delta_S + 2r^{-1}\Delta_S\Delta_{T^2} + r^{-2}\Delta_{T^2}^2,$$

where Δ_S is the ω_S-Laplacian on S and Δ_{T^2} is the ω_{T^2}-Laplacian on T^2.

Notice that the scalar curvature of g'_r is already constant, so that L'_r maps into functions with mean value zero. So, unlike the general case, there is no need to consider an additional projection. Separation of variables proves the following result.

Proposition 5.4. *Let $\psi \in C^\infty(S \times T^2)$ have mean value zero. There exists a unique function $\phi \in C^\infty(S \times T^2)$ with mean value zero solving the equation*

$$L'_r(\phi) = \psi. \tag{5.1}$$

There exists a constant C such that whenever ψ and ϕ have mean value zero and are related by equation (5.1),

$$\|\phi\|_{L^2(g'_r)} \leq Cr^2\|\psi\|_{L^2(g'_r)}.$$

Proof. Let v_λ be an $L^2(g_S)$-orthonormal basis of eigenvectors for Δ_S, with $\Delta_S v_\lambda = \lambda v_\lambda$. Let h_μ be an $L^2(g_{T^2})$-orthonormal basis of eigenvectors for Δ_{T^2}, with $\Delta_{T^2} h_\mu = \mu h_\mu$. Notice that $\lambda, \mu \geq 0$.

The functions $v_\lambda h_\mu$ are an $L^2(g'_1)$-orthonormal basis for functions over $S \times T^2$, and, in fact, are eigenfunctions for L'_r: $L'_r(v_\lambda h_\mu) = A_{\lambda\mu}(r) v_\lambda h_\mu$ where

$$A_{\lambda\mu}(r) = \left(\lambda^2 + \lambda\right) + 2r^{-1}\lambda\mu + r^{-2}\mu^2.$$

Notice that unless λ and μ are both zero, $A_{\lambda\mu}(r)$ is non-zero.

Given $\psi \in C^\infty(S \times T^2)$ write

$$\psi = \sum \psi_{\lambda\mu} v_\lambda h_\mu.$$

The condition that ψ has mean value zero is equivalent to $\psi_{00} = 0$. For such ψ define

$$\phi = \sum A_{\lambda\mu}(r)^{-1}\psi_{\lambda\mu}v_\lambda h_\mu. \tag{5.2}$$

where the sum is taken over λ and μ not both zero. Then $L'_r\phi = \psi$. Standard elliptic regularity results ensure that ϕ is smooth.

To prove the claim about the L^2-norms, begin with the $L^2(g'_1)$-norms. Notice that

$$\|\phi\|^2_{L^2(g'_1)} = \sum |A_{\lambda\mu}(r)|^{-2} |\psi_{\lambda\mu}|^2 .$$

Let μ_0 denote the smallest non-zero eigenvalue of Δ_{T^2}. For large r, the smallest value of $A_{\lambda\mu}(r)$ appearing in (5.2) is $A_{0\mu_0}(r) = r^{-2}\mu_0^2$. This means that

$$\|\phi\|_{L^2(g'_1)} \leq r^2\mu_0^{-2}\|\psi\|_{L^2(g'_1)}.$$

To prove the bound holds for $L^2(g'_r)$-norms, notice that for any L^2 function χ, $\|\chi\|_{L^2(g'_r)} = r^{1/2}\|\chi\|_{L^2(g'_1)}$. Hence the result follows from the bounds with $L^2(g'_1)$-norms. □

The above proof shows that L'_r has an inverse Q_r defined, at first, on smooth functions with mean value zero by equation (5.2):

$$Q_r(\psi) = \sum A_{\lambda\mu}(r)^{-1}\psi_{\lambda\mu}v_\lambda h_\mu.$$

Since $\|Q_r(\psi)\|_{L^2(g'_r)} \leq Cr^2\|\psi\|_{L^2(g'_r)}$, Q_r extends to a map $L^2 \to L^2$ on spaces of functions with integral zero. The following result will show that Q_r extends to a map $L^2_k \to L^2_{k+4}$, defined on functions of mean value zero, and that $\|Q_r\|$ remains $O(r^2)$ when considered in this way.

Lemma 5.5. *There exists a constant A such that for all $\phi \in L^2_{k+4}$,*

$$\|\phi\|_{L^2_{k+4}(g'_r)} \leq A\left(\|\phi\|_{L^2(g'_r)} + \|L'_r(\phi)\|_{L^2_k(g'_r)}\right).$$

Remark. For each r, the existence of such a constant follows from ellipticity of L'_r. The point of this result is that the constant can be chosen independently of r.

Proof. Recall the following standard result concerning interior estimates from the theory of elliptic operators. Let B be a compact Riemannian manifold (possibly with boundary), L a linear, order d elliptic differential operator on functions over B, and $B' \subset\subset B$ a subdomain. Then there exists a constant $a_{B,B',L}$ such that for all $\phi \in L^2_{k+4}(B)$,

$$\|\phi\|_{L^2_{k+d}(B')} \leq a_{B,B',L} \left(\|\phi\|_{L^2(B)} + \|L(\phi)\|_{L^2_k(B)} \right).$$

(See, for example, [Aub82].)

Consider the torus (T^2, rg_{T^2}) as the quotient of $\mathbb{R}^2$ by the lattice $r^{1/2}\mathbb{Z} \oplus r^{1/2}\mathbb{Z}$. Let m be the smallest integer larger than $r^{1/2}$. Cover $\mathbb{R}^2$ with discs of radius 1 with centres at points whose coordinates are integer multiples of $m^{-1}r^{1/2}$. This cover descends to a cover of T^2, by $O(r)$ discs, which in turn gives a cover of $S \times T^2$. Denote this cover by $\{B'_i\}$. Repeating the procedure, beginning with slightly larger discs of radius 1.01 with the same centres as before gives another cover $\{B_i\}$ of $S \times T^2$.

The above elliptic estimate applies to the operator L'_r over the pairs $B'_i \subset\subset B_i$. Since all such pairs are isometric and have geometry independent of r, the constant a is independent of i and r.

For any $\phi \in L^2_{k+4}(g'_r)$,

$$\begin{aligned}
\|\phi\|^2_{L^2_{k+4}(g'_r)} &\leq \sum_i \|\phi\|^2_{L^2_{k+4}(B'_i)}, \\
&\leq 2a^2 \sum_i \left(\|\phi\|^2_{L^2(B_i)} + \|L'_r(\phi)\|^2_{L^2_k(B_i)} \right), \\
&\leq 10a^2 \left(\|\phi\|^2_{L^2(g'_r)} + \|L'_r(\phi)\|^2_{L^2_k(g'_r)} \right), \\
&\leq 10a^2 \left(\|\phi\|_{L^2(g'_r)} + \|L'_r(\phi)\|_{L^2_k(g'_r)} \right)^2.
\end{aligned}$$

The extra factor of 5 comes from the fact that at most 5 of the B_i intersect at any one point. $\square$

Theorem 5.6. *The operator $L'_r \colon L^2_{k+4} \to L^2_k$ is a Banach space isomorphism, when considered between spaces of functions with mean value zero. There exists a constant C such that, for all large r, and all $\psi \in L^2_k$ with mean value zero, the inverse operator, Q_r, satisfies*

$$\|Q_r(\psi)\|_{L^2_{k+4}(g'_r)} \leq Cr^2 \|\psi\|_{L^2_k(g'_r)}.$$

Proof. All that remains to be proved is that the inverse Q_r, defined so far only on L^2 functions with mean value zero, extends to a linear operator $L^2_{k+4} \to L^2_k$ on functions with mean value zero, and is bounded as claimed.

This follows from the L^2-bounds (Proposition 5.4) and the uniform elliptic estimate (Lemma 5.5). For a smooth function ψ with mean value zero,

$$\begin{aligned} \|Q_r(\psi)\|_{L^2_{k+4}(g'_r)} &\leq A\left(\|Q_r(\psi)\|_{L^2(g'_r)} + \|L'_r Q_r(\psi)\|_{L^2_k(g'_r)}\right), \\ &\leq A(Cr^2+1)\|\psi\|_{L^2_k(g'_r)}. \end{aligned}$$

Hence Q_r extends and is bounded as claimed. □

5.2.2 Uniform Sobolev inequalities over a product

This section proves that certain Sobolev inequalities hold uniformly over a product, in preparation for proving the same result in general later on.

Lemma 5.7. *For indices k,l, and $q \geq p$ satisfying $k - 4/p \geq l - 4/q$ there is a constant c (depending only on p, q, k and l) such that for all $\phi \in L^p_k(g'_r)$,*

$$\|\phi\|_{L^q_l(g'_r)} \leq c\|\phi\|_{L^p_k(g'_r)}.$$

Remark. Again, for fixed r, the existence of such a c is a standard theorem concerning Sobolev spaces. The point of this result is that the constant can be taken independent of r. Notice that there are more stipulations on the indices (namely, $q \geq p$) than appear in the standard result. When this condition is not met, but $k - 4/p \geq l - 4/q$, there is still obviously a Sobolev inequality, but not a uniform constant. This is discussed later.

Proof. It is a standard result that, if B is any compact Riemannian manifold, possibly with boundary, there is a constant c_B such that for all functions $\phi \in L^q_l(B)$,

$$\|\phi\|_{L^q_l(B)} \leq c_B\|\phi\|_{L^p_k(B)}.$$

(See *e.g.* [Aub82].) Divide $(S \times T^2, g'_r)$ up into pieces $\{B_i\}$ as in the proof of Lemma 5.5. The above Sobolev inequality applies over each B_i, with c_B independent of i since all the pieces are isometric.

For any $\phi \in L^p_k(g'_r)$,

$$\begin{aligned} \|\phi\|^q_{L^q_l(g'_r)} &\leq \sum_i \|\phi\|^q_{L^q_l(B_i)}, \\ &\leq c^q_B \sum_i \|\phi\|^q_{L^p_k(B_i)}, \\ &\leq c^q_B \sum_i \left(\sum_{j=0}^k \int_{B_i} |\nabla^j \phi|^p\right)^{q/p}. \end{aligned}$$

Recall the following inequality: for positive a_i, and $m \geq 1$,

$$\sum_i a_i^m \leq \left(\sum_i a_i\right)^m.$$

Applying this to the inequality above it, and using the fact that $q \geq p$ gives

$$\|\phi\|^q_{L^q_l(g'_r)} \leq c_B^q \left(\sum_i \sum_{j=0}^k \int_{B_i} |\nabla^j \phi|^p\right)^{q/p} \leq 5c_B^q \|\phi\|^q_{L^p_k(g'_r)}.$$

which proves the result. (Again, the factor of 5 is due to at most of the B_i intersecting at any one point.) □

Lemma 5.8. *For indices p, k satisfying $k - 4/p > 0$, there is a constant c depending only on k and p, such that for all $\phi \in L^p_k(g'_r)$,*

$$\|\phi\|_{C^0} \leq c\|\phi\|_{L^p_k(g'_r)}.$$

Proof. From the standard theory (again, see [Aub82]), on any compact Riemannian manifold B, possibly with boundary, there exists a constant c_B such that for $\phi \in L^p_k(B)$ supported away from the boundary,

$$\|\phi\|_{C^0} \leq c_B\|\phi\|_{L^p_k(B)}.$$

For $x \in T^2$ let $\bar{D}(x,1)$ be a closed disc in T^2, centred at x and of radius 1 in the metric rg_{T^2}. Let $B_x = S \times \bar{D}(x,1)$. Let χ_x be the translates of a smooth cut-off function χ_0 supported in some $\bar{D}(x_0,1)$ which is 1 on a neighbourhood of x_0 and satisfies $0 \leq \chi_0 \leq 1$.

As each χ_x is essentially the same smooth function, there is a constant a, such that for any $\phi \in L^p_k(g'_r)$,

$$\|\chi_x\phi\|_{L^p_k(B_x)} \leq a\|\phi\|_{L^p_k(g'_r)}.$$

The functions $\chi_x\phi$ are supported away from the boundary in the isometric manifolds B_x Hence, by the result quoted above,

$$\begin{aligned}
\|\phi\|_{C^0} &= \sup_x \|\chi_x\phi\|_{C^0}, \\
&\leq \sup_x c_B\|\chi_x\phi\|_{L^p_k(B_x)}, \\
&\leq ac_B\|\phi\|_{L^p_k(g'_r)}.
\end{aligned}$$

□

5.2.3 Miscellaneous results

Two further lemmas concerning the product $(S \times T^2, g'_r)$ are needed for later developments.

Lemma 5.9. *For a tensor $\alpha \in C^k(T^{*\otimes i})$, as $r \to \infty$,*

$$\|\alpha\|_{C^k(g'_r)} = O(1).$$

Moreover, if α is pulled up from the base T^2,

$$\|\alpha\|_{C^k(g'_r)} = O\left(r^{-i/2}\right).$$

Proof. These statements follow from the fact that g'_r is a product metric: The Levi-Civita connection of g'_r depends only on those of g_S and rg_{T^2}, and hence is independent of r. The only r dependence in $C^k(g'_r)$, then, comes from the inner product on T^*. It is fixed for vertical (*i.e.* in the S directions) covectors and scales horizontal (*i.e.* in the T^2 directions) covectors by $r^{-1/2}$. This, and the fact that the horizontal-vertical splitting is g'_r-parallel, gives the result. □

Lemma 5.10. *There exists a constant C such that for any $u \in C^{k+4}(T^2)$, $\phi \in L^2_{k+4}(S \times T^2)$,*

$$\begin{aligned} \|L'_r(u\phi) - uL'_r(\phi)\|_{L^p_k(g'_r)} &\leq C\sum_{j=1}^{k+4} \|\nabla^j u\|_{C^0(g'_r)}\, \|\phi\|_{L^p_{k+4}(g'_r)}, \\ &= O\left(r^{-1/2}\right) \|\phi\|_{L^p_{k+4}(g'_r)}. \end{aligned}$$

Proof. This follows from the fact that the coefficients of L'_r are constant in the T^2 directions. The bound is $O(r^{-1/2})$ by Lemma 5.9, since all the derivatives of u are pulled back from T^2. □

5.2.4 Relationship with analysis over $S \times \mathbb{R}^2$

This section is a digression describing the link between analysis over $(S \times T^2, g'_r)$ and over $(S \times \mathbb{R}^2, g_S \oplus g_{\mathbb{R}^2})$, and the relationship between the results proved above and those already in the literature. The material here will not be used in the proof of Theorem 1.1.

First, notice that the above proofs of uniform Sobolev inequalities all involve cutting $(S \times T^2, g'_r)$ up into isometric pieces. Nowhere does it matter that this involves only finitely many pieces. The same arguments, then, which prove that a Sobolev inequality holds uniformly over $(S \times T^2, g'_r)$, show that the inequality also holds over the non-compact manifold $S \times \mathbb{R}^2$.

Conversely, if a Sobolev inequality holds over $S \times \mathbb{R}^2$ then it must hold uniformly over $(S \times T^2, g'_r)$. To see this, notice that a function over $(S \times T^2, g'_r)$ can be considered as a periodic function over $S \times \mathbb{R}^2$, with period $r^{1/2}$ along both axes in $\mathbb{R}^2$. Pick cut-off functions $f_r : \mathbb{R} \to \mathbb{R}$ which are 1 on $[0, r^{1/2}]$, supported in $[-1, r^{1/2}+1]$ and whose behaviour on $[r^{1/2}, r^{1/2}+1]$ and $[-1, 0]$ is independent of r. Define $\chi_r : \mathbb{R}^2 \to \mathbb{R}$ by $\chi_r(x, y) = f_r(x) f_r(y)$. Given $\phi \in L^p_k(g'_r)$, consider it as a periodic function over $S \times \mathbb{R}^2$, then $\chi_r \phi \in L^p_k(S \times \mathbb{R}^2)$.

There is a constant C such that

$$\|\phi\|_{L^p_k(g'_r)} \leq \|\chi_r \phi\|_{L^p_k(S \times \mathbb{R}^2)} \leq C \|\phi\|_{L^p_k(g'_r)}.$$

C is independent of r, since the behaviour of χ_r, outside of the fundamental domain, is essentially independent of r. From these inequalities it follows that any Sobolev inequality which holds over $S \times \mathbb{R}^2$ must hold uniformly over $(S \times T^2, g'_r)$.

Not all Sobolev inequalities which hold over $\mathbb{R}^4$ hold over $S \times \mathbb{R}^2$. As an example, consider the inequality

$$\|\phi\|_{L^q} \leq c \|\nabla \phi\|_{L^p},$$

where $p < 4$, and $q = 4p/(4 - p)$. This inequality holds for compactly supported functions over $\mathbb{R}^4$ (as is proved in, for example, [Aub82]).

To show that it cannot hold over $S \times \mathbb{R}^2$, consider the following functions. Let $\phi_R : \mathbb{R}^2 \to \mathbb{R}$ be radially symmetric, 1 on $D(0, R)$, supported in $D(0, R+1)$ and with behaviour for $R \leq |x| \leq R+1$ which is independent of R. Then, after pulling ϕ_R back to $S \times \mathbb{R}^2$,

$$\|\phi_R\|_{L^q(S \times \mathbb{R}^2)} = O(R^{2/q}),$$

whilst

$$\|\nabla \phi_R\|_{L^p(S \times \mathbb{R}^2)} = O(R^{1/p}).$$

Any p with $0 < p < 2$ gives $2/q > 1/p$. Hence, for these values of p, no such value of c exists.

Such examples can be constructed because of the close relationship between certain Sobolev inequalities and the isoperimetric inequality. With two of the four dimensions compactly "wrapped up" in $S \times \mathbb{R}^2$ there is sometimes a different interaction between the size of the boundary of a domain and the size of its interior than the dimension of the ambient space might suggest.

Finally, the analysis over $S \times \mathbb{R}^2$ which appears here is reminiscent of that which occurs in Floer theory over "tubes" $Y \times \mathbb{R}$ (where Y is a compact three-manifold). For this reason, the proofs given above mimic those in the literature on Floer theory. In particular the book [Don02a] has been followed closely here.

Local analysis 6

This chapter returns to the study of the approximate solutions $\omega_{r,n}$ constructed in Chapter 4. It proves that the product of a fixed curve with a flat torus, as considered in the previous section, is a local model for the Kähler structure $(X, J, \omega_{r,n})$ as $r \to \infty$. It then uses this model to prove various local analytic estimates (Sobolev inequalities, the elliptic estimate for $L_{r,n}$) hold uniformly in r.

The notation used here is all defined in Chapter 4. In particular the forms ω_0, ω_r and the function θ are defined in the proof of Lemma 4.1, whilst the higher order approximate solutions $\omega_{r,n}$ are constructed in Theorem 4.14.

6.1 Constructing the local model

Let $D \subset \Sigma$ be a holomorphic disc centred at σ_0. Since D is contractible, $X|_D$ is diffeomorphic to $S \times D$. The horizontal distribution of ω_0 is trivial when restricted to S_{σ_0}. By applying a further diffeomorphism if necessary the identification $X|_D \cong S \times D$ can be arranged so that the horizontal distribution on S_{σ_0} coincides with the restriction to S_{σ_0} of the TD summand in the splitting

$$T(S \times D) \cong TS \oplus TD. \tag{6.1}$$

For each value of r, there are two Kähler structures on $S \times D$ of immediate interest. The first comes from simply restricting the Kähler structure (X, J, ω_r) to $X|_D$. The complex structure has the form $J = J_\sigma \oplus J_D$ with respect to the splitting (6.1), where J_D is the complex structure on D and J_σ is the varying complex structure on the fibres.

The second is the natural product structure. With respect to (6.1), let

$$\begin{aligned} J' &= J_S \oplus J_D, \\ \omega'_r &= \omega_S \oplus r\omega_D. \end{aligned}$$

where ω_D is the flat Kähler form on D agreeing with ω_Σ at the origin, and J_S, ω_S are the complex structure and Kähler form on the central fibre $S = S_{\sigma_0}$. Denote by g'_r the corresponding metric on $S \times D$.

Notice that, by construction, the two complex structures agree on the central fibre, whilst the Kähler forms agree there except for the $\theta\omega_\Sigma$ term.

Theorem 6.1. *For all $\varepsilon > 0$, $\sigma_0 \in \Sigma$, there exists a holomorphic disc $D \subset \Sigma$, centred at σ_0, such that for all sufficiently large r, over $X|_D$,*

$$\|(J', \omega'_r) - (J, \omega_{r,n})\|_{C^k(g'_r)} < \varepsilon.$$

Proof. First notice that, by Lemma 5.9, for any holomorphic disc $D \subset \Sigma$, over $X|_D$,

$$\begin{aligned} \|\omega_{r,n} - \omega_r\|_{C^k(g'_r)} &\leq \sum_{i=1}^{n-1} \|i\bar{\partial}\partial r^{-i+1} f_i\|_{C^k(g'_r)} + \sum_{i=1}^{n} \|i\bar{\partial}\partial r^{-i}\phi_i\|_{C^k(g'_r)}, \\ &= O(r^{-1}). \end{aligned}$$

Since $\omega_{r,n} - \omega_r$ is $O(r^{-1})$ in $C^k(g'_r)$, it suffices to prove the result just for (J, ω_r).

Choose a holomorphic disc D centred at σ_0. The splitting (6.1) is parallel, implying that

$$\nabla^i(J - J') \in \mathrm{End}(TS) \otimes T^*(S \times D)^{\otimes i}.$$

The only changes in length as r varies come from the T^* factor. Write $\nabla^i(J - J') = \alpha_i + \beta_i$ with respect to the splitting

$$T^*(S \times D)^{\otimes i} \cong T^*S^{\otimes i} \oplus \Big(\big(T^*S^{\otimes i-1} \otimes T^*D\big) \oplus \cdots \oplus T^*D^{\otimes i}\Big).$$

$$\alpha_i \in T^*S^{\otimes i}, \qquad \beta_i \in \big(T^*S^{\otimes i-1} \otimes T^*D\big) \oplus \cdots \oplus T^*D^{\otimes i}.$$

The metric g'_r does not change in the S-directions, so $|\alpha_i|_{g'_r}$ is independent of r. Since $J = J'$ on the central fibre, reducing the size of D ensures that

$$|\alpha_i|_{g'_r} < \frac{\varepsilon}{2(k+1)}.$$

The metric g'_r scales lengths of cotangent vectors by $r^{-1/2}$ in the D-directions. So $|\beta_i|_{g'_r} = O(r^{-1/2})$. For large enough r,

$$|\beta_i|_{g'_r} < \frac{\varepsilon}{2(k+1)}.$$

Hence

$$\|\nabla^i(J - J')\|_{C^0(g'_r)} < \frac{\varepsilon}{k+1}.$$

Summing from $i = 0, \ldots k$ proves

$$\|J' - J\|_{C^k(g'_r)} < \varepsilon.$$

To prove $\|\omega'_r - \omega_r\|_{C^k(g'_r)} < \varepsilon$ it is enough to prove the same result for the metrics g_r, g'_r (since the Kähler forms can be recovered algebraically from the metric tensors via the complex structures).

Let u_1, u_2 be a local g_S-orthonormal frame for TS and v_1, v_2 be a local g_D-orthonormal frame for TD. Recall g_r induces a different horizontal-vertical splitting of the tangent bundle of $S \times D$, which is independent of r. With respect to this splitting,

$$g_r = g_\sigma \oplus (r + \theta) g_\Sigma,$$

where g_σ is the hyperbolic metric on the fibre S_σ, and g_Σ the metric on the base. Write $v_j = \eta_j + \xi_j$ with respect to the horizontal-vertical splitting induced by g_r (η_j is horizontal, ξ_j is vertical). With respect to the g'_r-orthonormal frame $u_1, u_2, r^{-1/2}v_1, r^{-1/2}v_2$, the matrix representative for g_r is

$$\begin{pmatrix} g_\sigma(u_i, u_j) & r^{-1/2} g_\sigma(u_i, \xi_j) \\ & \\ r^{-1/2} g_\sigma(u_j, \xi_i) & \left(1 + r^{-1}\theta\right) g_\Sigma(\eta_i, \eta_j) + r^{-1} g_\sigma(\xi_i, \xi_j) \end{pmatrix}$$

This means that, in a g'_r-orthonormal frame, $g_r - g'_r$ has the matrix representative

$$\begin{pmatrix} g_\sigma(u_i, u_j) - \delta_{ij} & 0 \\ & \\ 0 & g_\Sigma(\eta_i, \eta_j) - \delta_{ij} \end{pmatrix} + r^{-1/2} A + r^{-1} B.$$

for fixed matrices A and B.

The top left corner of the first term vanishes along the central fibre. Just as in the proof of $\|J' - J\| < \varepsilon$, this can be made arbitrarily small in $C^k(g'_r)$ by shrinking D and taking r large.

The bottom right corner of the first term is a function of the D-variables only. By construction it vanishes at the origin. The $C^0(g'_r)$-norm of this piece is just the conventional C^0-norm of the function $g_\Sigma(\eta_i, \eta_j) - \delta_{ij}$ and hence can be made arbitrarily small by shrinking D.

The derivatives of this piece are all in the D-directions. The length of the i-th derivative is $O(r^{-i/2})$ due to the scaling of g'_r in the D-directions. Hence the $C^k(g'_r)$ norm of this piece can be made arbitrarily small by taking r large (once D has been shrunk to deal with the C^0 term).

Finally, since A and B are independent of r, the $C^k(g'_r)$-norms of the tensors they represent are bounded as $r \to \infty$. So $r^{-1/2}A$, $r^{-1}B \to 0$ in $C^k(g'_r)$, which proves the theorem. $\square$

Whilst not particularly hard to prove (the only difficulties being notational!), Theorem 6.1 provides the essential local analytic input in the adiabatic limit. It justifies the statement that as $r \to \infty$ the local geometry is dominated by that of the fibre, and leads to the principle that local analytic results (such as the Sobolev inequalities of the previous chapter) which are true of the product are true of $(X, J, \omega_{r,n})$. There are several examples of this in what follows.

6.2 Uniform local analytic estimates

This section explains how to use the local model to prove that certain local analytic estimates hold over $(X, \omega_{r,n})$ uniformly in r.

6.2.1 Uniform equivalence of norms

Two different metrics g and g' on the same manifold may determine equivalent C^k- or L^p_k-norms. For example, suppose that

$$\|g - g'\|_{C^k(g')} \leq 1/2.$$

Then there exist positive constants b and B, such that for all functions ϕ,

$$b\|\phi\|' \leq \|\phi\| \leq B\|\phi\|'.$$

Here, $\|\cdot\|$ denotes either the C^k- or L^p_k-norm determined by g and $\|\cdot\|'$ the corresponding norm determined by g'.

To see this, first notice that the bound on $g - g'$ gives a $C^0(g')$-bound on the difference of the metrics on the cotangent bundle. This follows from the fact that the metric on covectors induced by g is, locally, given by g^{-1}. (It is here that it is necessary that $\|g - g'\|_{C^0(g')}$ be bounded by $1/2$ rather than some arbitrary constant.) It follows that the difference in induced metrics on any bundle of tensors is bounded in $C^0(g')$.

Next, recall the discussion in the proof of Lemma 3.8 where it is shown how a C^k bound on $g - g'$ gives a C^{k-1} bound on $\nabla - \nabla'$, the difference of the corresponding Levi-Civita connections. Combining this with the bound on the difference of pointwise norms shows C^k-norms defined by g and g' are equivalent. Since the difference in volume forms determined by g and g' is also bounded, the corresponding L^p_k-norms are also equivalent.

Notice, moreover, that the constants of equivalence found by this procedure do not depend on g or g', merely on the upper bound for $\|g' - g\|_{C^k(g')}$. One immediate application of this observation is to the behaviour of $g_{r,n}$-Banach space norms of tensors as $r \to \infty$.

Lemma 6.2. *For a tensor* $\alpha \in C^k(T^{*\otimes i})$*, as* $r \to \infty$*,*

$$\|\alpha\|_{C^k(g_{r,n})} = O(1).$$

Moreover, if α *is pulled up from the base,*

$$\|\alpha\|_{C^k(g_{r,n})} = O\left(r^{-i/2}\right).$$

Proof. By Lemma 5.9, the result is true for the local model. Let D be a disc over which Theorem 6.1 applies for $\varepsilon = 1/2$. Since $C^k(g_{r,n})$ and $C^k(g'_r)$ are uniformly equivalent over $X|_D$, the result holds for $C^k(g_{r,n})$ over $X|_D$. Cover Σ with finitely many discs D_i. The result holds for $C^k(g_{r,n})$ over each $X|_{D_i}$ and hence over all of X. □

This now gives a belated proof of Lemma 5.1, which describes the behaviour of $\|\operatorname{Scal}(\omega_{r,n})\|$ in $C^k(g_{r,n})$ and $L^2_k(g_{r,n})$ as $r \to \infty$.

Proof of Lemma 5.1. By the previous result, for any function ϕ, $\|\phi\|_{C^k(g_{r,n})}$ is bounded as $r \to \infty$. As is explained at the start of Section 5.1.1, this ensures that $\operatorname{Scal}(\omega_{r,n})$ is $O(r^{-n-1})$ in $C^k(g_{r,n})$.

To deduce the result concerning L^2_k-norms, notice that the g'_r-volume form is r times a fixed form. Hence, over a disc D where Theorem 6.1 applies, the $g_{r,n}$-volume form is $O(r)$ times a fixed form. So the volume of $X|_D$ with respect to $g_{r,n}$ is $O(r)$. Cover Σ with finitely many such discs, D_i. The volume $\operatorname{vol}_{r,n}$ of X, with respect to $g_{r,n}$, satisfies

$$\operatorname{vol}_{r,n} \leq \sum \operatorname{vol}(X|_{D_i}) = O(r).$$

Since, for smooth ϕ,

$$\|\phi\|_{L^2_k(g_{r,n})} \leq (\operatorname{vol}_{r,n})^{1/2} \|\phi\|_{C^k(g_{r,n})}$$

and since $\operatorname{vol}_{r,n}$ is $O(r)$, $\operatorname{Scal}(\omega_{r,n})$ is $O(r^{-n-1/2})$ in $L^2_k(g_{r,n})$. □

6.2.2 Patching and uniform estimates

To transfer other results from the product to X, a slightly more delicate patching argument is required. First the general set-up is described. Then it is applied to transfer the uniform estimates, proved over $S \times T^2$ in Section 5.2, to X.

The general setup

Set $\varepsilon = 1/2$ and cover Σ in discs $D_1, \dots, D_N$ satisfying the conclusions of Theorem 6.1. Let χ_i be a partition of unity subordinate to the cover D_i. Given a result which holds uniformly over the product, the partition of unity will be used to patch the local estimates together to obtain a global uniform estimate.

The first thing to control is the errors introduced by the partitions of unity. Let $\phi \in L^p_k$.

$$\|\chi_i \phi\|_{L^p_k(g_{r,n})} \le \|\chi_i\|_{C^k(g_{r,n})} \|\phi\|_{L^p_k(g_{r,n})}.$$

As a consequence of Lemma 6.2, $\|\chi_i\|_{C^k(g_{r,n})}$ is bounded as $r \to \infty$. Hence there exists a constant a such that, for any $i = 1, \dots, N$,

$$\|\chi_i \phi\|_{L^p_k(g_{r,n})} \le a \|\phi\|_{L^p_k(g_{r,n})}. \tag{6.2}$$

That is, the errors introduced by the patching can be controlled uniformly.

The local model $(S \times D, g_S \oplus r g_D)$ of Theorem 6.1 is an isometrically embedded submanifold of $(S \times T^2, g_S \oplus r g_{T^2})$, which was considered in Section 5.2. The function $\chi_i \phi$ is supported in $S \times D_i$ and so can be thought of as a function over $S \times T^2$. The results from Section 5.2 can then be applied.

Uniform estimates

Everything is now in place to transfer uniform estimates from $(S \times T^2, J', g'_r)$ to $(X, J, \omega_{r,n})$.

Lemma 6.3. *For indices k, l, and $q \ge p$ satisfying $k - 4/p \ge l - 4/q$ there is a constant c (depending only on p, q, k and l) such that for all $\phi \in L^p_k$ and all sufficiently large r,*

$$\|\phi\|_{L^q_l(g_{r,n})} \le c \|\phi\|_{L^p_k(g_{r,n})}.$$

Proof. Recall the analogous result for $(S \times T^2, g'_r)$ (Lemma 5.7). Using the partition of unity χ_i from above,

$$\begin{aligned} \|\phi\|_{L^q_l(g_{r,n})} &\le \sum \|\chi_i \phi\|_{L^q_l(g_{r,n})}, \\ &\le \text{const.} \sum \|\chi_i \phi\|_{L^q_l(g'_r)}. \end{aligned}$$

where the second inequality uses the uniform equivalence of the $g_{r,n}$- and g'_r-Sobolev norms.

Considering $\chi_i\phi$ as a function over $S \times T^2$, Lemma 5.7 gives

$$\|\chi_i\phi\|_{L^q_l(g'_r)} \leq \text{const.}\|\chi_i\phi\|_{L^p_k(g'_r)}.$$

Using the uniform equivalence of the $g_{r,n}$- and g'_r-Sobolev norms again gives

$$\|\chi_i\phi\|_{L^p_k(g'_r)} \leq \text{const.}\|\chi_i\phi\|_{L^p_k(g_{r,n})}.$$

Finally, combining these inequalities and inequality (6.2), which uniformly controls the errors caused by patching, gives

$$\|\phi\|_{L^q_l(g_{r,n})} \leq \text{const.}\sum\|\chi_i\phi\|_{L^p_k(g_{r,n})} \leq \text{const.}\|\phi\|_{L^p_k(g_{r,n})}.$$

□

Lemma 6.4. *For indices p, k satisfying $k - 4/p \geq 0$ there is a constant c, depending only on k and p, such that for all $\phi \in L^p_k$ and all sufficiently large r,*

$$\|\phi\|_{C^0} \leq c\|\phi\|_{L^p_k(g_{r,n})}.$$

Proof. Recall the analogous result for $(S \times T^2, g'_r)$ (Lemma 5.8). The same patching argument as above transfers the uniform estimate to $(X, J, \omega_{r,n})$. □

Lemma 6.5. *There is a constant A, depending only on k, such that for all $\phi \in L^2_{k+4}$ and all sufficiently large r,*

$$\|\phi\|_{L^2_{k+4}(g_{r,n})} \leq A\left(\|\phi\|_{L^2(g_{r,n})} + \|L_{r,n}(\phi)\|_{L^2_k(g_{r,n})}\right).$$

Proof. Recall the analogous result for L'_r over $(S \times T^2, g'_r)$ (Lemma 5.5). This time the patching argument must be combined with Lemma 3.11 which proves that the linearisation of the scalar curvature map is uniformly continuous with respect to the Kähler structure used to define it. To apply this result, it is necessary to observe that the curvature tensor of g'_r is bounded in $C^k(g'_r)$. Also, ε must be taken suitably small in Theorem 6.1.

Using the uniform equivalence of g'_r- and $g_{r,n}$-Sobolev norms, and with χ_i denoting a partition of unity subordinate to the cover D_i:

$$\begin{aligned}
\|\phi\|_{L^2_{k+4}(g_{r,n})} &\leq \text{const.}\sum\|\chi_i\phi\|_{L^2_{k+4}(g'_r)}, \\
&\leq \text{const.}\sum\left(\|\phi\|_{L^2(g'_r)} + \|L'_r(\chi_i\phi)\|_{L^2_k(g'_r)}\right),
\end{aligned}$$

where the last inequality uses Lemma 5.5.

Since the χ_i are functions on the base, by Lemma 5.10 (which describes the effect of pulling χ_i past L'_r),

$$\|L'_r(\chi_i\phi) - \chi_i L'_r(\phi)\|_{L^2_k(g'_r)} \leq \text{const}.r^{-1/2}\|\phi\|_{L^2_k(g'_r)}.$$

Using this, the uniform equivalence of g'_r- and $g_{r,n}$-Sobolev norms, and Lemma 3.11 to replace L'_r with $L_{r,n}$ gives

$$\|\phi\|_{L^2_{k+4}(g_{r,n})} \leq \text{const.}\left(\|\phi\|_{L^2(g_{r,n})} + \|\phi\|_{L^2_k(g'_r)} + \|L_{r,n}(\phi)\|_{L^2_k(g_{r,n})}\right).$$

This proves the result for $k = 0$. It also provides the inductive step giving the result for all k. □

Global analysis 7

The aim of this chapter is to prove the following result. (Recall that $L^2_{k,0}$ is the Sobolev space of functions with $g_{r,n}$-mean value zero, whilst p is projection onto such functions.)

Theorem 7.1. *For all large r and $n \geq 3$, the operator*

$$pL_{r,n} : L^2_{k+4,0} \to L^2_{k,0}$$

is a Banach space isomorphism. There exists a constant C, such that for all large r and all $\psi \in L^2_{k,0}$, the inverse operator $P_{r,n}$ satisfies

$$\|P_{r,n}\psi\|_{L^2_{k+4}(g_{r,n})} \leq Cr^3\|\psi\|_{L^2_k(g_{r,n})}.$$

The key point in Theorem 7.1 is the uniform control over $\|P_{r,n}\|$. Unlike the uniform Sobolev inequalities and elliptic estimate proved in the previous chapter, which are essentially local results, controlling the inverse $P_{r,n}$ of $pL_{r,n}$ is a global issue. Indeed it is only because of global considerations (compactness of X, no holomorphic vector fields) that such an inverse exists. This means that the local model used in the previous chapter is not directly useful. Instead a global model is used to make calculations more straightforward.

7.1 The global model

The approximate solutions $\pi\colon (X, \omega_{r,n}) \to \Sigma$ are not Riemannian submersions. The amount they differ from being so, however, tends to zero as r tends to infinity. This section uses this observation to relate the metrics $\omega_{r,n}$ to a family of Riemannian submersions which are easier to calculate with.

First recall that the form ω_0 (whose fibrewise restriction is the hyperbolic metric of that fibre) gives a vertical-horizontal decomposition of the tangent bundle of X. Define a Riemannian metric h_r on X by using the fibrewise metrics determined by ω_0 on the vertical vectors, and the metric $r\omega_\Sigma$ on the horizontal vectors.

By construction,

$$g_{r,0} = h_r + a$$

for some tensor $a \in \Gamma(T^*X \otimes T^*X)$, independent of r, which is essentially given by the horizontal components of ω_0.

Since T^*X is scaled by $r^{-1/2}$ in the metric h_r it follows immediately that for all r sufficiently large,

$$\|g_{r,0} - h_r\|_{C^0(h_r)} < 1/2. \tag{7.1}$$

Moreover, since

$$\|g_{r,n} - g_{r,0}\|_{C^0(h_r)} = O\left(r^{-1}\right),$$

inequality (7.1) holds with $g_{r,0}$ replaced by $g_{r,n}$. In particular this means that the L^2-norms on tensors determined by h_r and $g_{r,n}$ are equivalent with constants of equivalence independent of r:

Lemma 7.2. *Let E denote any bundle of tensors. There exist positive constants k and K such that for all $t \in \Gamma(E)$ and all sufficiently large r,*

$$k\|t\|_{L^2(h_r)} \leq \|t\|_{L^2(g_{r,n})} \leq K\|t\|_{L^2(h_r)}.$$

7.2 The lowest eigenvalue of $\mathscr{D}^*\mathscr{D}$

It is more convenient to work first with the positive self adjoint elliptic operator $\mathscr{D}^*\mathscr{D}$. Here $\mathscr{D} = \bar{\partial} \circ \nabla$ where $\bar{\partial}$ is the $\bar{\partial}$-operator of the holomorphic tangent bundle and $\mathscr{D}^*$ is the L^2-adjoint of $\mathscr{D}$. Notice that $\mathscr{D}^*\mathscr{D}$ depends on $\omega_{r,n}$, and so on r (and n). This section finds a lower bound for its first non-zero eigenvalue. First, however, its kernel is described.

Lemma 7.3. *There are no nonzero holomorphic vector fields on X.*

Proof. Recall that the fibres and base of X have high genus. The short exact sequence of holomorphic bundles

$$0 \to V \to TX \to \pi^*T\Sigma \to 0$$

gives a long exact sequence in cohomology

$$0 \to H^0(X, V) \to H^0(X, TX) \to H^0(X, \pi^*T\Sigma) \to \cdots$$

Hence it suffices to prove that $H^0(X, V)$ and $H^0(X, \pi^*T\Sigma)$ both vanish.

$H^0(X, V) = 0$ as the fibres admit no nonzero holomorphic vector fields. Since π is a submersion with compact fibres, $\pi_*\pi^*T\Sigma = T\Sigma$. Composing π with the projection $\Sigma \to$ pt. shows that $\pi^*T\Sigma$ has the same space of global sections as $T\Sigma$. So $H^0(X; \pi^*T\Sigma) = 0$ also. □

Corollary 7.4. $\ker \mathscr{D}^*\mathscr{D} = \mathbb{R}$. *Equivalently,*

$$\mathscr{D}^*\mathscr{D} \colon L^2_{k+4,0} \to L^2_{k,0}$$

is an isomorphism.

Proof. $\ker \mathscr{D}^*\mathscr{D} = \ker \mathscr{D}$ is those functions with holomorphic gradient. The previous lemma implies such functions must be constant. The second statement now follows from the fact that $\mathscr{D}^*\mathscr{D}$ is an elliptic formally self adjoint index zero operator. □

To find a lower bound for the first non-zero eigenvalue of $\mathscr{D}^*\mathscr{D}$, similar bounds are first found for the Hodge Laplacian and for the $\bar{\partial}$-Laplacian on sections of the holomorphic tangent bundle.

Lemma 7.5. *There exists a positive constant C_1 such that for all ϕ with $g_{r,n}$-mean value zero and for all sufficiently large r,*

$$\|\mathrm{d}\phi\|^2_{L^2(g_{r,n})} \geq C_1 r^{-1} \|\phi\|^2_{L^2(g_{r,n})}.$$

Remark. That this inequality corresponds to a lower bound on the first eigenvalue of Δ follows from the existence of a complete L^2-orthonormal basis of eigenvectors for Δ, with the kernel spanned by constants.

Proof. Using Lemma 7.2,

$$\|\mathrm{d}\phi\|_{L^2(g_{r,n})} \geq \text{const.}\, \|\mathrm{d}\phi\|_{L^2(h_r)}.$$

There exists a constant m such that $\phi - m$ has h_1-mean value zero. (In fact, as h_r is a Riemannian submersion scaled by r in the horizontal directions, the condition of having h_r-mean value zero is independent of r.) Since m is constant, $\mathrm{d}\phi = \mathrm{d}(\phi - m)$.

Let $|\cdot|_{h_r}$ denote the pointwise inner product defined by h_r. By definition of h_r it follows that

$$\begin{aligned} |\mathrm{d}(\phi - m)|^2_{h_r} &= |\mathrm{d}_V(\phi - m)|^2_{h_r} + |\mathrm{d}_H(\phi - m)|^2_{h_r}, \\ &= |\mathrm{d}_V(\phi - m)|^2_{h_1} + r^{-1} |\mathrm{d}_H(\phi - m)|^2_{h_1}, \\ &\geq r^{-1} |\mathrm{d}(\phi - m)|^2_{h_1}, \end{aligned}$$

where d_V, d_H denote the vertical and horizontal components of d. Moreover, the volume forms satisfy $\mathrm{dvol}(h_r) = r\mathrm{dvol}(h_1)$. Hence,

$$\|\mathrm{d}(\phi - m)\|^2_{L^2(h_r)} \geq \|\mathrm{d}(\phi - m)\|^2_{L^2(h_1)}.$$

Now $\phi - m$ has h_1-mean value zero. Let c be the first eigenvalue of the h_1-Laplacian. Then

$$\begin{aligned} \|\mathrm{d}(\phi - m)\|^2_{L^2(h_1)} &\geq c\, \|\phi - m\|^2_{L^2(h_1)}, \\ &= c r^{-1} \|\phi - m\|^2_{L^2(h_r)}. \end{aligned}$$

Using Lemma 7.2 again gives

$$\begin{aligned} \|\phi - m\|^2_{L^2(h_r)} &\geq \text{const.} \|\phi - m\|^2_{L^2(g_{r,n})} \\ &\geq \text{const.} \|\phi\|^2_{L^2(g_{r,n})} \end{aligned}$$

where the second inequality follows from the fact that ϕ has $g_{r,n}$-mean value zero.

Putting the pieces together shows that there exists a positive constant C_1 such that

$$\|\mathrm{d}\phi\|_{L^2(g_{r,n})} \geq C_1 r^{-1} \|\phi\|^2_{L^2(g_{r,n})}$$

as required. □

Lemma 7.6. *There exists a positive constant C_2 such that for all $\xi \in \Gamma(TX)$ and for all sufficiently large r,*

$$\left\|\bar{\partial}\xi\right\|^2_{L^2(g_{r,n})} \geq C_2 r^{-2} \|\xi\|^2_{L^2(g_{r,n})} .$$

Remark. Notice that, since $\ker \bar{\partial} = 0$ (Lemma 7.3 proves that X admits no holomorphic vector fields), there is no need to impose the condition $\xi \in (\ker \bar{\partial})^{\perp}$ analogous to that in the previous result where ϕ was required to have $g_{r,n}$-mean value zero.

Proof. The proof is similar to that of Lemma 7.5 above. By Lemma 7.2,

$$\left\|\bar{\partial}\xi\right\|^2_{L^2(g_{r,n})} \geq \text{const.} \left\|\bar{\partial}\xi\right\|^2_{L^2(h_r)} .$$

Notice that $\bar{\partial}\xi \in \Gamma(T^*X \otimes TX)$ has four components under the splitting induced by the vertical-horizontal decomposition of TX. Considering the behaviour of h_r on these components gives

$$\left|\bar{\partial}\xi\right|^2_{h_r} \geq r^{-1} \left|\bar{\partial}\xi\right|^2_{h_1} . \tag{7.2}$$

Using this and the relationship $\mathrm{dvol}(h_r) = r\mathrm{dvol}(h_1)$ gives

$$\left\|\bar{\partial}\xi\right\|^2_{L^2(h_r)} \geq \left\|\bar{\partial}\xi\right\|^2_{L^2(h_1)} .$$

Let c be the first eigenvalue of the $\bar{\partial}$-Laplacian determined by the metric h_1. Then

$$\left\|\bar{\partial}\xi\right\|^2_{L^2(h_1)} \geq c \|\xi\|^2_{L^2(h_1)} .$$

Considering the behaviour of h_r on the horizontal and vertical components of ξ shows that

$$|\xi|^2_{h_1} \geq r^{-1} |\xi|^2_{h_r}.$$

Hence,

$$\|\xi\|^2_{L^2(h_1)} \geq r^{-2} \|\xi\|^2_{L^2(h_r)}$$

Finally, using Lemma 7.2 to convert back to the $L^2(g_{r,n})$-norm of ξ, and putting all the pieces together shows that there exists a positive constant C_2 such that

$$\left\|\bar{\partial}\xi\right\|^2_{L^2(g_{r,n})} \geq C_2 r^{-2} \|\xi\|^2_{L^2(g_{r,n})}$$

as required. □

Lemma 7.7. *There exists a constant C such that whenever ϕ has $g_{r,n}$-mean value zero and r is sufficiently large,*

$$\|\mathscr{D}\phi\|^2_{L^2(g_{r,n})} \geq C r^{-3} \|\phi\|^2_{L^2(g_{r,n})}.$$

Proof. Combining Lemmas 7.5 and 7.6 shows that whenever ϕ has $g_{r,n}$-mean value zero,

$$\begin{aligned}
\left\|\bar{\partial}\nabla\phi\right\|^2_{L^2(g_{r,n})} &\geq C_2 r^{-2} \|\nabla\phi\|^2_{L^2(g_{r,n})}, \\
&= C_2 r^{-2} \|\mathrm{d}\phi\|^2_{L^2(g_{r,n})}, \\
&\geq C_1 C_2 r^{-3} \|\phi\|^2_{L^2(g_{r,n})}.
\end{aligned}$$

□

7.3 A uniformly controlled inverse

The section proves that (for all large r and $n \geq 3$) $pL_{r,n}$ is invertible between spaces of functions with mean value zero. It also converts the lower bound for the first non-zero eigenvalue of $\mathscr{D}^*\mathscr{D}$ into an upper bound for the norm of the inverse $P_{r,n}$ of $pL_{r,n}$.

Lemma 7.8. *There is a constant A, depending only on k, such that for all $\phi \in L^2_{k+4}$ and for all sufficiently large r,*

$$\|\phi\|_{L^2_{k+4}(g_{r,n})} \leq A \left(\|\phi\|_{L^2(g_{r,n})} + \|\mathscr{D}^*\mathscr{D}(\phi)\|_{L^2_k(g_{r,n})} \right).$$

Proof. Recall equation (3.3):

$$L_{r,n}(\phi) = \mathscr{D}^*\mathscr{D}(\phi) + \nabla\,\mathrm{Scal}(\omega_{r,n})\cdot\nabla\phi$$

Since $\mathrm{Scal}(\omega_{r,n}) \to 0$ in $C^k(g_{r,n})$, $L_{r,n} - \mathscr{D}^*\mathscr{D}$ converges to zero in operator norm calculated with respect to the $L^2_k(g_{r,n})$-Sobolev norms. Hence the estimate follows from the analogous result for $L_{r,n}$ (Lemma 6.5). $\square$

Theorem 7.9. *The operator*

$$\mathscr{D}^*\mathscr{D} : L^2_{k+4,0} \to L^2_{k,0}$$

is a Banach space isomorphism. There exists a constant K, such that for all large r and all $\psi \in L^2_{k,0}$, the inverse operator Q_r satisfies

$$\|Q_r\psi\|_{L^2_{k+4}(g_{r,n})} \le Kr^3\|\psi\|_{L^2_k(g_{r,n})}.$$

Proof. The inverse Q_r exists by Corollary 7.4. It follows from the lower bound on the first non-zero eigenvalue of $\mathscr{D}^*\mathscr{D}$ (Lemma 7.7 applied to $\phi = Q_r(\psi)$) that there is a constant C such that for all $\psi \in L^2_{k,0}$,

$$\|Q_r(\psi)\|_{L^2(g_{r,n})} \le Cr^3\|\psi\|_{L^2(g_{r,n})}.$$

Applying Lemma 7.8 to $\phi = Q_r\psi$ extends this bound to the one required. $\square$

Before converting this into a proof of Theorem 7.1, a lemma is proved showing that invertibility is an open condition.

Lemma 7.10. *Let $D\colon B_1 \to B_2$ be an bounded invertible linear map of Banach spaces with bounded inverse Q. If $L\colon B_1 \to B_2$ is another linear map with*

$$\|L - D\| \le (2\|Q\|)^{-1},$$

then L is also invertible with bounded inverse P satisfying

$$\|P\| \le 2\|Q\|.$$

Proof. By assumption,

$$\|(LQ-1)\| \le \frac{1}{2}$$

Hence $\sum(LQ-1)^j$ converges to an operator R satisfying $LQR = 1$, $\|R\| \le 2$. So $P = QR$ is a right inverse for L with $\|P\| \le 2\|Q\|$. Repeat the argument with $1 - QL$ to find a left inverse. This completes the proof. $\square$

The pieces are now in place to prove Theorem 7.1 which is stated at the start of this chapter.

Proof of Theorem 7.1. Since

$$(L_{r,n} - \mathscr{D}^*\mathscr{D})\phi = \nabla \operatorname{Scal}(\omega_{r,n}) \cdot \nabla\phi,$$

there exists a constant c such that, in operator norm computed with respect to the $g_{r,n}$-Sobolev norms,

$$\|pL_{r,n} - \mathscr{D}^*\mathscr{D}\| \leq cr^{-n-1}.$$

So for $n \geq 3$, and for large enough r,

$$\|pL_{r,n} - \mathscr{D}^*\mathscr{D}\| \leq (2\|Q_r\|)^{-1}.$$

Lemma 7.10 shows that $pL_{r,n}$ is invertible and gives the upper bound

$$\|P_{r,n}\| \leq 2\|Q_r\| \leq Cr^3$$

for some C. $\square$

7.4 An improved bound

It should be possible to improve on this estimate. For example, Theorem 5.6 shows that, over a Kähler product,

$$\|P_r\| \leq Cr^2.$$

The above proof of Theorem 7.1 concatenates two eigenvalue estimates, each of which is saturated only when applied to an eigenvector corresponding to the first eigenvalue. The functions which get closest to saturating the first estimate (Lemma 7.5) have gradients which can be controlled more efficiently than is done in the proof of the second estimate (Lemma 7.6).

For example, in the product case, the first eigenspace of the Laplacian consists, for large r, of functions on the base. These functions have horizontal gradients. The splitting of the tangent bundle into horizontal and vertical components is, over the product at least, a holomorphic splitting. So, if ϕ is a function on the base, then $\bar{\partial}\nabla\phi$ has purely horizontal vector and covector factors. This means that the estimate (7.2) appearing in the proof of Lemma 7.6 can be improved, for $\xi = \nabla\phi$, to

$$\left|\bar{\partial}\nabla\phi\right|^2_{h_r} = \left|\bar{\partial}\nabla\phi\right|^2_{h_1},$$

thus gaining a power of r.

In general, it should be possible to obtain a better bound for $\|P_{r,n}\|$ by examining this interplay between the two eigenvalue estimates. An alternative way to improve the bound might be to consider $\mathscr{D}$ as a whole, rather than factor by factor. The additional complication that appears here is that $\mathscr{D} = \bar{\partial}\nabla$ depends on the metric via ∇, unlike the operators $\bar{\partial}$ and d considered above. This metric dependence could be dealt with by estimating the resulting r dependence. However, the bound proved above is sufficient to complete the proof of Theorem 1.1. This is done in the next chapter.

Completing the proof 8

The previous chapter proves that the linearisation $pL_{r,n}$ of $S_{r,n}$ is an isomorphism with inverse $P_{r,n}$ which is $O(r^3)$. The final step needed to apply the inverse function theorem is to control the nonlinear operator $S_{r,n} - pL_{r,n}$.

8.1 Controlling the nonlinear terms

Denote by $\mathrm{Scal}_{r,n}$ the scalar curvature map on Kähler potentials determined by $\omega_{r,n}$:

$$\mathrm{Scal}_{r,n}(\phi) = \mathrm{Scal}(\omega_{r,n} + i\bar{\partial}\partial\phi)$$

Recall that $S_{r,n} = p\,\mathrm{Scal}_{r,n}$. Denote by $N_{r,n} = S_{r,n} - pL_{r,n}$ the nonlinear terms of $S_{r,n}$.

Lemma 8.1. *Let $k \geq 3$. There exists positive constants c and K, such that for all ϕ, $\psi \in L^2_{k+4}$ with $\|\phi\|_{L^2_{k+4}}$, $\|\psi\|_{L^2_{k+4}} \leq c$ and all r sufficiently large,*

$$\|N_{r,n}(\phi) - N_{r,n}(\psi)\|_{L^2_k} \leq K \max\left\{\|\phi\|_{L^2_{k+4}},\ \|\psi\|_{L^2_{k+4}}\right\} \|\phi - \psi\|_{L^2_{k+4}}$$

where $g_{r,n}$-Sobolev norms are used throughout.

Proof. By the mean value theorem,

$$\|N_{r,n}(\phi) - N_{r,n}(\psi)\|_{L^2_k(g_{r,n})} \leq \sup_{\chi\in[\phi,\psi]} \|(DN_{r,n})_\chi\| \|\phi - \psi\|_{L^2_{k+4}(g_{r,n})} \tag{8.1}$$

where $(DN_{r,n})_\chi$ is the derivative of $N_{r,n}$ at χ.

Now

$$DN_{r,n} = p(L_{r,n})_\chi - pL_{r,n}$$

where $(L_{r,n})_\chi$ is the linearisation of $\mathrm{Scal}_{r,n}$ at χ. In other words, $(L_{r,n})_\chi$ is the linearisation of the scalar curvature map determined by the metric $\omega_{r,n} + i\bar{\partial}\partial\chi$. Since

$$\|\chi\|_{L^2_{k+4}} \leq \max\left\{\|\phi\|_{L^2_{k+4}},\ \|\psi\|_{L^2_{k+4}}\right\}, \tag{8.2}$$

for a suitable choice of $c > 0$ Lemma 3.12 can be applied to the metrics $\omega_{r,n}$ and $\omega_{r,n} + i\bar{\partial}\partial\chi$. As $k \geq 3$ the condition on the indices in Lemma 3.12 is met. Notice also that Lemma 3.12 requires the constants in the $g_{r,n}$-Sobolev inequalities to be uniformly bounded — which is proved in Lemmas 6.3 and 6.4 — and the $C^k(g_{r,n})$-norm of the curvature of $\omega_{r,n}$ to be bounded above — which follows from Theorem 6.1 and Lemma 3.8.

The outcome is that

$$\|(L_{r,n})_\chi - L_{r,n}\| \leq \mathrm{const.}\|\chi\|_{L^2_{k+4}(g_{r,n})}.$$

The map p is uniformly bounded (an $L^2_k(g_{r,n})$-orthogonal projection even); hence

$$\|(DN_{r,n})_\chi\| \leq \text{const.}\|\chi\|_{L^2_{k+4}(g_{r,n})}.$$

The result now follows from this and inequalities (8.1) and (8.2). ☐

8.2 Applying the inverse function theorem

The pieces are finally in place to apply the inverse function theorem.

Theorem 8.2. *For all sufficiently large r and $k \geq 3$, there exists $\phi \in L^2_{k+4}$ with* $\mathrm{Scal}(\omega_r + i\bar{\partial}\partial\phi)$ *constant.*

Proof. For all sufficiently large r, and k, $n \geq 3$ the map $S_{r,n}\colon L^2_{k+4}(g_{r,n}) \to L^2_k(g_{r,n})$ has the following properties:

1. $S_{n,r}(0) = O\left(r^{-n-1/2}\right)$ in $L^2_k(g_{r,n})$, by Lemma 5.1.
2. The derivative of $S_{r,n}$ at the origin is an isomorphism with inverse $P_{r,n}$ which is $O(r^3)$. This is proved in Theorem 7.1.
3. There exists a constant K such that for all sufficiently small M, the nonlinear piece $N_{r,n}$ of $S_{r,n}$ is Lipschitz with constant M on a ball of radius KM. This follows directly from Lemma 8.1.

Recall the statement of the inverse function theorem (Theorem 5.3). The second and third of the above properties imply that the radius $\delta'_{r,n}$ of the ball about the origin on which $N_{r,n}$ is Lipschitz with constant $(2\|P_{r,n}\|)^{-1}$ satisfies

$$\delta'_{r,n} \geq \text{const.}\, r^{-3}$$

for some positive constant and all large r. As $\delta_{r,n} = \delta'_{r,n}(2\|P_{r,n}\|)^{-1}$ it follows that

$$\delta_{r,n} \geq \text{const.}\, r^{-6}$$

for some positive constant and all large r.

Hence for $\psi \in L^2_k$ with

$$\|S_{r,n}(0) - \psi\|_{L^2_k(g_{r,n})} \leq \text{const.}\, r^{-6}$$

the equation $S_{r,n}(\phi) = \psi$ has a solution. In particular, the first of the above properties implies that, for $n \geq 6$ and all large r, the equation $S_{r,n}(\phi) = 0$ has a solution. Since $\mathrm{Scal}(\omega_{r,n} + i\bar{\partial}\partial\phi)$ differs from $S_{r,n}(\phi)$ by a constant, the result follows. ☐

8.3 Regularity

The final step is to check that the solution obtained via the inverse function theorem is in fact smooth.

Lemma 8.3. *Let (X, ω) be a Kähler manifold and S the scalar curvature map on Kähler potentials. Let $k \geq 2$. If $\phi \in C^{k,\alpha}$ satisfies $S(\phi) \in C^{k,\alpha}$ then $\phi \in C^{k+4,\alpha}$.*

Proof. The function $S(\phi)$ is defined by $S(\phi) = \Delta_\phi V$ where Δ_ϕ is the Laplacian of the metric $\omega + i\bar{\partial}\partial\phi$ and $V = -\log\det(g + \Phi)$ with Φ being the symmetric tensor associated to the real $(1,1)$-form $i\bar{\partial}\partial\phi$.

Since $\phi \in C^{k,\alpha}$, Δ_ϕ is a linear second order elliptic operator with coefficients in $C^{k-2,\alpha}$. By elliptic regularity (see *e.g.* [Aub82], page 87) and assumption on $S(\phi)$, $V \in C^{k,\alpha}$.

The map $\phi \mapsto -\log\det(g + \Phi)$ is nonlinear, second order and elliptic. Such maps also satisfy a regularity result (see *e.g.* [Aub82], page 86); since $V \in C^{k,\alpha}$, this gives $\phi \in C^{k+2,\alpha}$. This in turn implies that Δ_ϕ has $C^{k,\alpha}$ coefficients meaning that, in fact, $V \in C^{k+2,\alpha}$ and so $\phi \in C^{k+4,\alpha}$. $\square$

By Theorem 8.2, for all $k \geq 3$, there exists $\phi \in L^2_{k+4}$ with $\mathrm{Scal}(\omega_r + i\bar{\partial}\partial\phi)$ constant. For k high enough, it follows from the Sobolev embedding theorem that $L^2_{k+4} \hookrightarrow C^{2,\alpha}$. Iteratively applying the previous regularity result gives $\phi \in C^{l,\alpha}$ for all l and so ϕ is smooth.

This completes the proof of Theorem 1.1.

Extensions *9*

This chapter discusses two possible extensions of Theorem 1.1 and the difficulties involved in proving them.

9.1 Higher dimensional varieties

The most obvious generalisation of Theorem 1.1 is to consider higher dimensional fibrations $\pi \colon X \to B$ where X is a compact connected complex manifold, and π is a holomorphic submersion. Bearing in mind the summary at the end of Chapter 4, the following is a list of conditions on $\pi \colon X \to B$ under which the same arguments used to prove Theorem 1.1 may prove the existence of constant scalar curvature metrics:

1. Let X be a Kähler manifold with no nonzero holomorphic vector fields. Let $\kappa_0 \in H^{1,1}(X)$ be a Kähler class; its restriction to each fibre $F_b = \pi^{-1}(b)$, denoted κ_b, is a Kähler class for that fibre.

2. For every $b \in B$, assume that the class κ_b contains a unique constant scalar curvature metric ω_b; moreover, assume that ω_b depends smoothly on b. Let ω be a Kähler form representing κ_0; then for each b there is a unique function ϕ_b on F_b with mean value zero such that the fibrewise restriction of $\omega + i\bar{\partial}\partial\phi_b = \omega_b$. The smoothness assumption ensures that the ϕ_b fit together to give a smooth function ϕ on X, so $\omega_0 = \omega + i\bar{\partial}\partial\phi$ is a Kähler form whose fibrewise restriction has constant scalar curvature.

3. The metrics ω_b give a Hermitian structure in the vertical cotangent bundle V^* and hence also in the line bundle $\Lambda^{max} V^*$. Denote its curvature F. Taking the fibrewise mean value of the horizontal-horizontal component of iF (with respect to the metric ω_0 described above) defines a form $\alpha \in \Omega^{1,1}(B)$. Assume that there is a metric ω_B on the base solving the equation $\mathrm{Scal}(\omega_B) - \Lambda\alpha = \text{const.}$; moreover, assume that there are no nontrivial deformations of ω_B through cohomologous solutions to this equation.

The proof of the following should be very similar to that of Theorem 1.1.

Conjecture 9.1. *Under the above conditions, for all large r, the Kähler class*

$$\kappa_r = \kappa_0 + r[\omega_B]$$

contains a constant scalar curvature Kähler metric.

It should also be possible to relax the condition that X have no nonzero holomorphic vector fields. This is only needed to ensure the weaker condition that just constant functions have holomorphic gradient.

There is a more unsatisfactory aspect about the above list of conditions. They involve solving a strange partial differential equation on the base. Not only is it unfamiliar, it depends on the whole manifold X, and not just on B. The conditions would seem more natural if 3 could be replaced by the condition that the base admitted a constant scalar curvature metric, with no nontrivial cohomologous deformations through constant scalar curvature metrics.

Indeed, from the algebro-geometric viewpoint (and assuming the conjectural equivalence between stably polarised varieties, and constant scalar curvature metrics discussed in Chapter 1), the conditions would then correspond to X being a family of stable varieties parametrised by a stable variety. These are conditions under which the stability of X could reasonably be expected, at least with respect to a polarisation corresponding to κ_r. One reason to believe this is that the analogous statement is true for holomorphic bundles: a family of stable bundles over (F, ω_F) parametrised by B gives, for all large r, a stable bundle over $(F \times B, \omega_F \oplus r\omega_B)$.

This suggests that perhaps it should be possible to prove the existence of constant scalar curvature Kähler metrics with condition 3 above replaced by the following.

3′. Assume that there is a Kähler metric ω_B with constant scalar curvature. Assume, moreover, that there is no nontrivial infinitesimal deformation of ω_B through cohomologous constant scalar curvature Kähler metrics.

Conjecture 9.2. *Let X satisfy conditions 1, 2, and 3′ above. Then, for all large r, the Kähler class*

$$\kappa_r = \kappa_0 + r[\omega_B]$$

contains a constant scalar curvature Kähler metric.

Alternatively condition 3 may itself have an algebro-geometric interpretation. Namely, X may be stable with respect to the polarisation $\kappa_0 + r[\omega_B]$ if the fibres are stably polarised by the restriction of κ_0 and if the base is also stably polarised, not with respect to $[\omega_B]$, but rather some other polarisation constructed from $[\omega_B]$ and the push down of the vertical tangent bundle of X.

9.2 Holomorphic Lefschetz fibrations

A holomorphic Lefschetz fibration on a compact connected complex surface X is a holomorphic surjection $\pi\colon X \to \mathbb{P}^1$ which is a submersion away from a finite number of points p_k. Moreover, there are holomorphic coordinates centred at each p_k in which the map π has the form

$$(z, w) \mapsto zw$$

Away from the p_k, X is locally biholomorphic (preserving π) to the surfaces under consideration in this thesis. At each p_k, however, the fibres develop nodal singularities.

A strong generalisation of Theorem 1.1 would be to include holomorphic Lefschetz fibrations with generic fibres of genus at least 2. This would greatly increase the number of surfaces covered by the result. Indeed any projective surface is birational to a holomorphic Lefschetz fibration. The changes needed in the proof, however, are substantial. Even the approximate solutions fail to go over easily: the fibrewise hyperbolic metrics become singular at the nodes.

To rectify this the metric may be adjusted on a neighbourhood of each node. The key step will be to find a metric on the model $(z, w) \mapsto zw$ which has zero scalar curvature and is asymptotic, in the correct sense, to the singular metric on X (*i.e.* the metric constructed from the fibrewise hyperbolic metrics and a large multiple of the metric on the base). Scaling this metric by a small parameter ε means that interpolation between the singular metric on X and the model metric near a node can be done over smaller and smaller neighbourhoods. Since the local model is scalar-flat, its scalar curvature does not blow up during this rescaling. This gives a family of nonsingular metrics depending on parameters r and ε. If the asymptotics of the local model are correct, the scalar curvature of this metric should approach minus one as $r \to \infty$ and $\varepsilon \to 0$.

Finding the correct asymptotics and model metric will be hard problems. It may be possible to use the toric symmetry of $(z, w) \mapsto zw$ to simplify matters. Abreu [Abr03] and Guillemin [Gui94] demonstrate that calculations are much more straightforward in the toric case.

Also, a similar problem is considered in part of [GW00]. The authors there are concerned with elliptic K3 surfaces. An approximation to the Ricci-flat metric on such a surface is obtained by gluing together a metric on the smooth part of the fibration which is Ricci-flat on the fibres, with a model "Ooguri-Vafa" metric near the singular fibres. This construction is very close to that outlined above and studying it closely would certainly be beneficial.

There are other problems in converting the proof of Theorem 1.1 to apply to holomorphic Lefschetz fibrations besides finding an approximate solution. It is perhaps wisest, however, to find the approximate solution before worrying about perturbing it.

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