Chapter 2

Holomorphic line bundles

In the absence of non-constant holomorphic functions $X \to \mathbb{C}$ on a compact complex manifold, we turn to the next best thing, holomorphic sections of line bundles (i.e., rank one holomorphic vector bundles).

In this section we explain how Hermitian holomorphic line bundles carry a natural connection and hence one can talk of the curvature of such a bundle. From here we define the first Chern class, the main topological invariant of a line bundle. We then turn to the question of prescribing the curvature of a holomorphic line bundle and explain how this can always be done on a Kähler manifold. We close with a description of the correspondence between line bundles (at least those which admit meromorphic sections) and divisors (i.e., finite linear combinations of analytic hypersurfaces).

2.1 First Chern class via Chern–Weil

We begin with some general facts about holomorphic vector bundles. Recall that a complex vector bundle $E \to X$ over a complex manifold is holomorphic if there are local trivialisations $E|_{U_\alpha} \cong \mathbb{C}^k \times U_\alpha$ for which the transition functions $\phi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{C})$ are *holomorphic*. In such a situation we can define a $\bar\partial$-operator on sections

$$\bar\partial \colon C^\infty(E) \to \Omega^{0,1}(E)$$

where $\Omega^{0,1}(E)$ denotes smooth sections of $\Lambda^{0,1} \otimes E$. To define $\bar\partial$, first look in a local trivialisation over U_α. Here sections are given by functions $U_\alpha \to \mathbb{C}^k$ and we know how to take $\bar\partial$ of such a thing. Now when we change trivialisations we do so by a holomorphic map and so $\bar\partial$ is unaffected, meaning the local definitions patch together to give an operator defined on global sections as claimed.

We now compare $\bar\partial$-operators and connections.

Definition 2.1. Let X be a complex manifold and $E \to X$ a holomorphic vector bundle. A connection ∇ in E is said to be *compatible with the holomorphic structure in* E if $\pi^{0,1}(\nabla s) = \bar{\partial} s$ for all sections s of E.

Proposition 2.2. *Let E be a Hermitian holomorphic vector bundle. Then there is a unique connection in E compatible with both the Hermitian and holomorphic structures.*

Definition 2.3. The distinguished connection in Proposition 2.2 is called the *Chern connection*.

Proof of Proposition 2.2 for line bundles. We prove this for a line bundle $L \to X$. (The proof for vector bundles of higher rank is left as an exercise.) In a local holomorphic trivialisation, connections compatible with the holomorphic structure have the form $\nabla^A = \mathrm{d} + A$ where A is a $(1,0)$-form. Meanwhile, the Hermitian structure h is given in the trivialisation by a smooth real-valued positive function, which we continue to denote h. The condition $\nabla^A h = 0$ amounts to $Ah + h\bar{A} = \mathrm{d}h$ which, when combined with the fact that A is of type $(1,0)$, gives $A = \partial \log h$. It follows that there is a unique choice of A such that ∇^A is compatible with both structures. We can do this in each local trivialisation of L; by uniqueness the a priori locally defined Chern connections all agree over intersections and so give a globally defined connection. □

Whilst different metrics will certainly lead to different Chern connections there is a topological restriction on the possible curvatures.

Lemma 2.4. *Given a Hermitian metric h in a holomorphic line bundle $L \to X$, the curvature F_h of the Chern connection is closed and, moreover, $\frac{i}{2\pi}[F_h] \in H^2(M, \mathbb{R})$ is independent of the choice of h.*

Proof. The curvature of L in the local trivialisation is given by $F_h = dA = \bar{\partial}\partial \log h$ and so F_h is certainly closed. Note moreover that $iF_h = i\bar{\partial}\partial \log h$ is a real $(1,1)$-form and so $i[F_h] \in H^2(M, \mathbb{R})$ as claimed.

To prove that it is independent of the metric, write $h' = e^f h$ for a second Hermitian metric in L, where f is any smooth function $X \to \mathbb{R}$. The corresponding curvatures are related by $F_{h'} = F_h + \bar{\partial}\partial f$. Since $\bar{\partial}\partial f = \mathrm{d}\partial f$ it follows that the cohomology class $[F_h]$ is independent of the choice of metric h and depends only on the holomorphic line bundle L. □

Definition 2.5. We write $c_1(L) = \frac{i}{2\pi}[F_h] \in H^2(X, \mathbb{R})$ where h is any Hermitian metric in L. This is called the *first Chern class of L*.

This is actually a simple example of a much more general phenomenon in algebraic topology. The approach to the first Chern class we have taken here is direct but contains only what we will need in the rest of these notes. What is not apparent from our brief discussion is that :

- The class $c_1(L) \in H^2(X, \mathbb{R})$ is actually the image of a class in $H^2(X, \mathbb{Z})$. This lift is what is more normally known as the first Chern class of L. (Notice that the de Rham class will vanish if the integral class is torsion, so the integral class carries strictly more information.)
- In fact, one can use the same definition for *any* unitary connection in L with respect to any Hermitian metric, not just one compatible with the holomorphic structure.
- It follows that the first Chern class depends only on the topological isomorphism class of $L \to X$ (and not its holomorphic structure).
- These classes can be defined without using connections at all, for line bundles over any sufficiently nice topological (e.g., topological manifolds).
- One can define higher Chern classes for vector bundles of higher rank in a similar fashion by constructing differential forms out of their curvature tensors. Again, this gives an image in de Rham cohomology of the genuine topological invariants which live in integral cohomology. And again, this can all be done without connections on topological manifolds.

We will not pursue these matters here.

Of course, for this discussion to be of interest, one must have some holomorphic line bundles in the first place. There is always one holomorphic line bundle you are guaranteed to have to hand:

Definition 2.6. Let X be a complex manifold. The holomorphic line bundle $K = \Lambda^n(T^*X)$ is called the *canonical line bundle* and its dual K^* the *anti-canonical line bundle*. The *first Chern class of* X is defined by $c_1(X) = c_1(K^*) = -c_1(K)$.

Exercises 2.7.

1. Prove Proposition 2.2 by following the same proof as was given above for line bundles.
2. Let $L \to \mathbb{C}^n$ be the trivial bundle with Hermitian metric $h = e^{-|z|^2}$. Compute the curvature of the corresponding Chern connection.
3. Let $L \to \mathbb{C}$ be the trivial bundle with Hermitian metric $h = 1 + |z|^2$. Compute the curvature F of the corresponding Chern connection.

   Calculate $\int_{\mathbb{C}} F$.
4. Given line bundles L_1, L_2, prove that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.
5. Given a vector bundle E, we can define $c_1(E) = c_1(\det E)$ where $\det E$ is the top exterior power of E.

(a) Prove for vector bundles E_1, E_2 that $c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2)$.

(b) Prove that if L is a line bundle and E a vector bundle of rank r then $c_1(L \otimes E) = rc_1(L) + c_1(E)$.

2.2 Line bundles over projective space

We next consider holomorphic line bundles over complex projective space. There is a tautological bundle over $\mathbb{CP}^n$, denoted $\mathcal{O}(-1)$ (for reasons which will soon become clear). A point $x \in \mathbb{CP}^n$ corresponds to a line $L_x < \mathbb{C}^{n+1}$; the fibre of $\mathcal{O}(-1)$ over x is precisely L_x.

The dual of $\mathcal{O}(-1)$ is denoted $\mathcal{O}(1)$ and is called the *hyperplane bundle*. More generally, for $k \in \mathbb{Z}$, $\mathcal{O}(k)$ denotes the k^{th}-power of the hyperplane bundle (if k is negative then we interpret this as the $-k^{\text{th}}$-power of $\mathcal{O}(-1)$).

We can write down holomorphic sections of $\mathcal{O}(1)$ in a simple fashion. Let $\alpha \in (\mathbb{C}^{n+1})^*$. Note that the fibre L_x of $\mathcal{O}(-1)$ at x is a linear subspace of $\mathbb{C}^{n+1}$ and so, by restriction, α defines a linear map $L_x \to \mathbb{C}$. In other words, α determines a section s_α of $\mathcal{O}(1)$.

We can now explain the reason for the name "hyperplane bundle". The linear map $\alpha \colon \mathbb{C}^{n+1} \to \mathbb{C}$ has kernel a hyperplane in $\mathbb{C}^{n+1}$ which descends to a hyperplane $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$. Tracing through the definition we see that this is exactly the zero locus of s_α.

The above construction shows that the space of holomorphic sections of $\mathcal{O}(1)$ has dimension at least $n+1$. In fact, the s_α account for all the holomorphic sections of $\mathcal{O}(1)$ as we will prove later.

We next turn to sections of $\mathcal{O}(k)$ for $k > 1$. To write some down, we can take tensor products of sections of $\mathcal{O}(1)$, giving sections of the form $s_{\alpha_1}^{p_1} \otimes s_{\alpha_j}^{p_j}$ where $p_1 + \cdots + p_j = k$. Another way to interpret such sections is to consider a homogeneous degree k polynomial giving a map $\mathbb{C}^{n+1} \to \mathbb{C}$. Restricting to L_x we get a degree k map $L_x \to \mathbb{C}$ or, equivalently a linear map $L_x^k \to \mathbb{C}$ and so a section of $\mathcal{O}(k)$. Again we will prove later that the homogeneous degree k polynomials account for all holomorphic sections of $\mathcal{O}(k)$.

Exercises 2.8.

1. Check that the bundle $\mathcal{O}(-1)$ defined above is genuinely a holomorphic line bundle.

   Check that s_α defined above is genuinely a holomorphic section of $\mathcal{O}(1)$.

2. Fix a Hermitian metric on $\mathbb{C}^{n+1}$. Since the fibre L_x of $\mathcal{O}(-1)$ at x is a subspace of $\mathbb{C}^{n+1}$ the Hermitian structure induces a Hermitian metric in the line bundle

$\mathcal{O}(-1)$. Compute the curvature of the corresponding Chern connection. How does it relate to the Fubini–Study metric?

2.3 Prescribing the curvature of a line bundle

We have seen that when $L \to X$ is a holomorphic Hermitian line bundle, its curvature gives a real $(1,1)$-form $\frac{i}{2\pi}F$ representing $c_1(L)$. A natural question is: given a $(1,1)$-form $\Phi \in -2\pi i c_1(L)$ is there a Hermitian metric h in L with $F_h = \Phi$?

Fix a reference metric h_0. Then $h = e^f h_0$ is the metric we seek if and only if f solves

$$\bar{\partial}\partial f = \Phi - F_{h_0}.$$

This question is the basic prototype of more difficult questions which we will encounter later.

On a Kähler manifold, it turns out one can always solve the above question, thanks to the following Lemma.

Lemma 2.9 (The $\bar{\partial}\partial$-lemma)**.** *Let (X, J, ω) be a compact Kähler manifold and let α_1, α_2 be cohomologous real $(1,1)$-forms. Then there exists $\phi\colon X \to \mathbb{R}$ such that $\alpha_1 = \alpha_2 + i\bar{\partial}\partial\phi$. Such a function ϕ is unique up to the addition of a constant.*

The proof is in the exercises.

Corollary 2.10. *Given a holomorphic line bundle $L \to X$ over a Kähler manifold and a real $(1,1)$-form $\Phi \in -2\pi i c_1(L)$ there is a unique Hermitian metric h, up to constant scale, with $F_h = \Phi$.*

The $\bar{\partial}\partial$-lemma also gives an extremely convenient description of all Kähler forms in a fixed cohomology class.

Corollary 2.11. *If ω_1, ω_2 are two Kähler metrics in the same cohomology class then there exists a smooth function ϕ, unique up to addition of a constant, such that $\omega_1 = \omega_2 + i\bar{\partial}\partial\phi$.*

Definition 2.12. Given two cohomologous Kähler metrics ω_1, ω_2 a function ϕ satisfying $\omega_1 = \omega_2 + i\bar{\partial}\partial\phi$ is called the *Kähler potential* of ω_1 relative to ω_2.

If one has locally, $\omega = i\bar{\partial}\partial\phi$, then ϕ is called a *local Kähler potential for ω*.

This simple fact is one of the most important reasons why Kähler metrics are more tractable than general Riemannian metrics: the metric is determined by a single scalar function, rather than a matrix valued function.

Exercises 2.13. Let X be a compact Kähler manifold. Prove the $\bar{\partial}\partial$-lemma as follows.

1. Let θ be a $(0,1)$-form.

   (a) Prove that there is a function $u\colon X \to \mathbb{C}$ such that $\bar{\partial}^*(\theta - \bar{\partial}u) = 0$.

   (b) Prove, moreover, that if $\bar{\partial}\theta = 0$ then with this same choice of u we have $\partial(\theta - \bar{\partial}u) = 0$.

2. Let $\alpha = \mathrm{d}\beta$ be a real $(1,1)$-form.

   (a) By applying the results of the previous step to $\theta = \beta^{0,1}$, prove that there is a (complex valued) function u for which $\partial\beta = -\bar{\partial}\partial u$.

   (b) Deduce that $\alpha = i\bar{\partial}\partial\phi$ for a real-valued function ϕ.

   (c) Prove moreover that ϕ is unique up to the addition of a constant.

2.4 Line bundles and divisors

Let $L \to X$ be a holomorphic line bundle and s a holomorphic section. At least when s vanishes transversely, its zero locus is a smooth complex hypersurface, i.e., a complex submanifold of codimension 1. The language of divisors enables one to associate a linear combination of hypersurfaces, called a divisor, to holomorphic (or meromorphic) sections which don't necessarily vanish transversely. Moreover, it turns out that one can recover the line bundle L from the divisor of a meromorphic section. In this way we will set up a correspondence between divisors (modulo linear equivalence) and line bundles which admit meromorphic sections.

We begin with the definition of a divisor.

Definition 2.14. Let X be a complex manifold.

1. An *analytic hypersurface* $V \subset X$, or hypersurface for short, is a subset which can locally be written as the zero locus of a single holomorphic function. I.e., there is an open cover $\{U_\alpha\}$ of X and holomorphic functions $f_\alpha\colon U_\alpha \to \mathbb{C}$ such that $V \cap U_\alpha = f_\alpha^{-1}(0)$.

2. A hypersurface is called *irreducible* if it cannot be written as the union of two non-empty hypersurfaces and, moreover, the local defining functions f_α are chosen to vanish to order 1. Otherwise it is called *reducible*.

3. A *divisor* is a finite formal sum $D = \sum m_j V_j$ of irreducible hypersurfaces V_j with integer coefficients m_j.

4. The set $\mathrm{Div}(X)$ of divisors is a group under addition in the obvious way.

5. A divisor $\sum m_j V_j$ is called *effective* if $m_j \geq 0$ for all j. (We allow $m_j = 0$ to include the zero divisor.) We write $D \geq 0$ to indicate that D is effective.

Next we describe how to associate a divisor to a meromorphic section of a holomorphic line bundle $L \to X$. (A meromorphic section is one which is given in a local trivialisation by a meromorphic function, equivalently by a locally defined holomorphic function with values in $\mathbb{CP}^1$.) To do this we need to describe the order of a zero and of a pole of such a section.

Definition 2.15. Let s be a meromorphic section of $L \to X$.

1. Suppose that $s(x) = 0$ and choose a local trivialisation of L over a coordinate chart U of x, in which $s \colon U \to \mathbb{C}$ is regarded as a meromorphic function on a subset of $\mathbb{C}^n$. The *order of vanishing of s at x* is the lowest $m \in \mathbb{N}$ such that s has a non-zero partial derivative of order m at x.

2. Suppose that s has a pole at x. Again choose a local trivialisation of L in a chart near x in which s is regarded as a meromorphic function. We say *s has a pole of order m at x* if the meromorphic function $1/s$ has a zero of order m at x.

One should check that this definition does not depend on the choice of local trivialisation and chart which are used to define the partial derivatives.

Definition 2.16. Let s be a meromorphic section of a holomorphic line bundle.

1. The *zero divisor of s* is the formal sum
$$Z(s) = \sum m_j V_j$$
where the V_j are the irreducible components of $s^{-1}(0)$ and $m_j \in \mathbb{N}$ is the order of vanishing of s along V_j.

2. The *polar divisor of s* is the formal sum
$$P(s) = \sum n_j W_j$$
where the W_j are the irreducible components of $s^{-1}(\infty)$ and $n_j \in \mathbb{N}$ is the order of the pole of s along W_j.

3. The *divisor of s* is the formal sum
$$\operatorname{div}(s) = Z(s) - P(s)$$

Note that s is holomorphic if and only if $\operatorname{div}(s)$ is effective.

Example 2.17. Consider the polynomial $s_\epsilon(x, y, z) = xy - \epsilon z^2$ As explained above this corresponds to a section of $\mathcal{O}(2) \to \mathbb{CP}^2$. When $\epsilon \neq 0$, the zero locus of s_ϵ is a conic C_ϵ, the image of a non-linear embedding of $\mathbb{CP}^1$ inside $\mathbb{CP}^2$. When $\epsilon = 0$, the zero locus is a pair of linearly embedded $\mathbb{CP}^1$s given by $\{x = 0\}$ and $\{y = 0\}$. When

$\epsilon \to \infty$, the conic C_ϵ collapses to two copies of the same linearly embedded $\mathbb{CP}^1$ given by $\{z = 0\}$. (This can be seen by looking at the rescaled polynomial $\epsilon^{-1}xy - z^2$.)

$$\operatorname{div}(s_\epsilon) = \begin{cases} C_\epsilon & \text{for } \epsilon \neq 0 \\ \mathbb{P}(x=0) + \mathbb{P}(y=0) & \text{for } \epsilon = 0 \\ 2\mathbb{P}(z=0) & \text{for } \epsilon = \infty \end{cases}$$

We next explain how to associate a holomorphic line bundle to a divisor. We begin with an irreducible hypersurface $V \subset X$. Write $\{U_\alpha\}$ for a cover of X with functions $f_\alpha \colon U_\alpha \to \mathbb{C}$ which vanish to order 1 along $V \cap U_\alpha$. On an intersection $U_{\alpha\beta}$ the quotient $f_{\alpha\beta} = f_\alpha / f_\beta$ is holomorphic and non-vanishing, since the zeros of the numerator and denominator cancel exactly. Moreover, $f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha} = 1$. This means we can use the $f_{\alpha\beta}$ as transition functions for a line bundle which we denote L_V. Finally, notice that L_V comes with a holomorphic section: since $f_\alpha = f_{\alpha\beta} f_\beta$ the f_αs are the local representatives of a holomorphic section which we denote s_V.

Definition 2.18. Given a divisor $D = m_1 V_1 + \cdots + m_k V_k$, the *line bundle of D* is

$$L_D = L_{V_1}^{m_1} \otimes \cdots \otimes L_{V_k}^{m_k}$$

where if $m < 0$, then L^m means $(L^*)^m$.

L_D also comes with a *meromorphic* section. To see this, return to the case of an irreducible hypersurface as discussed above. The transition functions for L_V^* are $f_{\alpha\beta}^{-1}$. This means that the functions f_α^{-1} define a meromorphic section s_V^{-1} of L_V^*. We can now write

$$s_D = s_{V_1}^{m_1} \otimes \cdots \otimes s_{V_k}^{m_k}$$

for the distinguished meromorphic section of L_D.

The multiplicative notation $L^{-1} = L^*$ is no accident. Line bundles up to isomorphism form an abelian group, with the trivial bundle as identity and the dual as the inverse.

Definition 2.19. Denote by $\operatorname{Pic}(X)$ the set of isomorphism classes of holomorphic line bundles. Tensor product makes $\operatorname{Pic}(X)$ into an abelian group, called *the Picard group of X*.

In proving that $\operatorname{Pic}(X)$ is a group, the main thing to check is that $L \otimes L^*$ is trivial. To see this, identify $L \otimes L^* \cong \operatorname{End}(L)$; the endomorphism bundle has a nowhere vanishing section given by the identity $L \to L$; since $\operatorname{End}(E)$ is also of rank 1 this means it is trivial.

Proposition 2.20.

1. *The map $D \mapsto L_D$ gives a homomorphism* $\operatorname{Div}(X) \to \operatorname{Pic}(X)$.

2. *The divisor D can be recovered from (L_D, s_D) via $D = \mathrm{div}(s_D)$.*

3. *Given a line bundle L with a meromorphic section s there is an isomorphism $L_{\mathrm{div}(s)} \cong L$.*

4. *The kernel of the homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is those divisors of the form $\mathrm{div}(f)$ where f is a meromorphic function on X.*

5. *The image of $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is those line bundles admitting a meromorphic section.*

Proof. The first two points follow immediately from the definitions. To prove the third point, that $L_{\mathrm{div}(s)} \cong L$ note that $L_{\mathrm{div}(s)}$ has a meromorphic section t with the same divisor as s. In other words, $s^{-1} \otimes t$ is a holomorphic nowhere vanishing section of $L^* \otimes L_D$ giving the required isomorphism.

To prove the fourth point, suppose that L_D is isomorphic to the trivial bundle. Write "1" for the nowhere vanishing holomorphic section of L_D arising from a trivialisation $L_D \cong \mathcal{O}$. The meromorphic section s_D is of the form $f \cdot 1$ for some meromorphic function f and we have $D = \mathrm{div}(s_D) = \mathrm{div}(f)$. Conversely, if $D = \mathrm{div}(f)$ then, regarding f as a section of the trivial bundle, we see from the third part that $L_D \cong \mathcal{O}$.

Finally, the fifth point follows from the above. □

Definition 2.21. Two divisors D, D' are called *linearly equivalent* if $L_D \cong L_{D'}$ which is the same as saying $D - D' = \mathrm{div}(f)$ for some meromorphic function f.

We will see later that when (X, ω) is Kähler and $[\omega]$ is actually the first Chern class $c_1(L)$ of some line bundle, then in fact all holomorphic line bundles over X admit meromorphic sections and so the map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ is actually surjective, giving an isomorphism $\mathrm{Pic}(X) \cong \mathrm{Div}(X)/\sim$ where $\sim$ denotes linear equivalence.

Lemma 2.22. *If two divisors are linearly equivalent then they are homologous. (When talking of the homology class of $D = \sum m_j V_j$, we mean $[D] = \sum m_j [V_j] \in H_{2n-2}(X, \mathbb{Z})$.)*

Proof. Let s_D and $s_{D'}$ be meromorphic sections of a line bundle L corresponding to two linearly equivalent divisors D, D'. The section $s_t = (1-t)s_D + t s_{D'}$ gives a path of divisors giving a homology from D to D'. □

In fact we can do much better than this Lemma: the homology class of D is Poincaré dual to $c_1(L_D)$. This is a consequence of the Poincaré–Lelong formula, which we do not prove.

Theorem 2.23 (Poincaré–Lelong)**.** *Let $D = \sum m_j V_j \in \mathrm{Div}(X)$. Then $c_1(L_D) \in H^2(X, \mathbb{Z})$ is Poincaré dual to the homology class $[D]$*

When $V \subset X$ is a smooth irreducible hypersurface there is a succinct geometry interpretation of L_V. Recall that the normal bundle of V is the line bundle $N \to V$ given by taking the quotient of $TX|_V$ by TV.

Proposition 2.24. *Given a smooth irreducible hypersurface V, there is an isomorphism $N \cong L_V|_V$ between the normal bundle of V and the restriction of L_V to V.*

Proof. Write N^* for the conormal bundle. The dual of the surjection $TX|_V \to N$ gives an injection $N^* \to T^*X|_V$, with image the sub-bundle of 1-forms which vanish on vectors tangent to V.

Now let $\{U_\alpha, f_\alpha\}$ be an open cover of X with locally defining functions f_α for V. The transition functions of L_V are $f_{\alpha\beta} = f_\alpha / f_\beta$. At points of $V \cap U_\alpha \cap U_\beta$, the locally defined holomorphic 1-forms $\mathrm{d}f_\alpha$ satisfy $\mathrm{d}f_\alpha = f_{\alpha\beta}\mathrm{d}f_\beta$. This means that the $\mathrm{d}f_\alpha$ fit together to give a global nowhere vanishing section of $T^*X|_V \otimes L$. Moreover, since V is irreducible, $\mathrm{d}f_\alpha$ is non-zero on V. Finally, since f_α vanishes on V, $\mathrm{d}f_\alpha$ vanishes on vectors tangent to V. This means that they combine to give a nowhere vanishing section of $N * \otimes L_V|_V$ or, in other words, an isomorphism $N \to L_V|_V$. □

The following is a useful corollary of this result, which is often called the adjunction formula.

Corollary 2.25. *[Adjunction formula] Let $V \subset X$ be a smooth irreducible hypersurface. Then the canonical bundles of V and X are related by*

$$K_V \cong K_X|_V \otimes L_V^{-1}|_V$$

The proof is an exercise.

2.5 Linear systems and maps to projective space

Consider two independent sections s, s' of a holomorphic line bundle $L \to X$. Given $[p, q] \in \mathbb{CP}^1$ we define the divisor $D[p, q] = \mathrm{div}(ps + qs')$. (Note that whilst the section $ps + qs'$ depends on the pair (p, q) of coordinates, the divisor depends only on the image in $\mathbb{CP}^1$.) The divisors $D[p, q]$ are all linearly equivalent, since they come from sections of the same bundle L.

We can use s, s' to define a map to $\mathbb{CP}^1$ which is defined on almost all of $\mathbb{CP}^2$. To do this, suppose that s and s' do not both vanish at $[x, y, z]$. Then we can find a unique point $[p, q] \in \mathbb{CP}^1$ such that $(ps + qs')(x, y, z) = 0$. In other words, exactly one member of the family of divisors passes through $[x, y, z]$. If we let $B = s^{-1}(0) \cap s'^{-1}(0)$, then $f[x, y, z] = [p, q]$ defines a map $f \colon X \setminus B \to \mathbb{CP}^1$ whose fibres are the divisors in our family.

We can generalise this construction to larger families of holomorphic sections. We begin with a holomorphic line bundle $L \to X$ and a linear subspace $V \subset H^0(X, L)$ of holomorphic sections. Such a V, called a *linear system*, determines a map to projective space in the following way.

Let $s_0, \dots, s_d$ be a basis of V and define the map $f\colon X \to \mathbb{CP}^d$ by

$$f(x) = [s_0(x) : \cdots : s_d(x)]$$

There are three things to mention here.

Firstly, the $s_j(x)$ are not, as the notation here suggests, genuine complex numbers, rather they are all elements in the same complex line L_x, the fibre of L over $x \in X$. In order to make sense of the above expression, one must first choose an isomorphism $L_x \cong \mathbb{C}$, under which the $s_j(x) \in L_x$ are now identified with complex numbers $s'_j(x) \in \mathbb{C}$ say. The point is that if one chooses a different isomorphism between L_x and $\mathbb{C}$, the $s_j(x)$ become identified with different elements $s''_j(x) \in \mathbb{C}$ *but* since the two different identifications of L_x with $\mathbb{C}$ differ simply by multiplication by some $\alpha \in \mathbb{C} \setminus 0$, these new elements are related to the old ones by $s''_j(x) = \alpha s'_j(x)$ for all j and hence the corresponding point in projective space is unchanged. This is what is meant by the above map.

Secondly, it is possible that f is not defined at all points of X, namely if all sections in V vanish at some x, then f will not be defined there. This leads to the following definition.

Definition 2.26. Given a holomorphic line bundle $L \to X$ and a subspace $V \subset H^0(X, L)$, the set B of common zeros of sections of V is called the *base locus* of V. Given a basis $s_0, \dots, s_d$ of V, there is a well defined map $f\colon X \setminus B \to \mathbb{CP}^d$, called the *map corresponding to the linear system* V. When $B = \varnothing$, one says that V is base point free.

Thirdly, we can interpret this in terms of the family of divisors defined by the sections $s \in V$. Given a hyperplane $H \subset \mathbb{CP}^d$, then $f^{-1}(H) \cup B$ is a divisor in X of the form $s^{-1}(0) \setminus B$ for a unique $s \in V$.

Example 2.27. Recall that a holomorphic section of $\mathcal{O}(k)$ is a homogeneous polynomial of degree k in $d+1$ variables. It can be checked that space of such polynomials has dimension $N_{k,d} = \frac{(k+d)!}{k!d!}$. Since there is no point of $\mathbb{C}^{d+1}$ at which all such polynomials vanish, the base locus of the complete linear system is empty and we get a map $\mathbb{CP}^d \to \mathbb{CP}^{N_{k,d}-1}$, called *the Veronese embedding*. It is not difficult to check that this is indeed an embedding.

There is a more invariant way of defining the map f which does not involve the choice of a basis. Evaluation at a point $x \in X$ defines a linear map $\mathrm{ev}_x\colon V \to L_x$. Picking an identification $L_x \cong \mathbb{C}$ we identify ev_x with an element in V^*. Changing the identification $L_x \cong \mathbb{C}$ scales this element of V^* by a non-zero constant and so, at least assuming ev_x is not identically zero, we obtain a well-defined element of $\mathbb{P}(V^*)$.

Definition 2.28. Given a holomorphic line bundle $L \to X$ and a subspace $V \subset H^0(X, L)$ there is a canonically defined map, $f\colon X \setminus B \to \mathbb{P}(V^*)$, called the *map corresponding to the linear system* V.

At least away from the base locus, we can recover L from the map f.

Lemma 2.29. *Given a line bundle $L \to X$, a linear system V with base locus B and map $f\colon X \setminus B \to \mathbb{P}(V^*)$, there is a natural identification between $L|_{X\setminus B}$ and the pullback $f^*\mathcal{O}(1)$ of the hyperplane bundle.*

The proof is an exercise.

Exercises 2.30.

1. Check that Definition 2.15 does not depend on the choice of local trivialisation used to define the partial derivatives of s.

2. (a) Let $D \in \mathrm{Div}(X)$. Prove that there is an isomorphism between the space of holomorphic sections of L_D and the space of meromorphic functions f for which $D + \mathrm{div}(f) \geq 0$.

   (b) Prove that if $D > 0$ then L_D^* has no holomorphic sections.

3. Let $D \in \mathrm{Div}(\mathbb{CP}^n)$ be an effective divisor which is linearly equivalent to a hyperplane $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$.

   (a) Prove that if $x, y \in D$ (i.e., x, y are points lying in the irreducible hypersurfaces which make up D) then the linear $\mathbb{CP}^1$ joining x and y is contained in D. *Hint: consider the intersection number* $[\mathbb{CP}^1] \cdot [D]$.

   (b) Prove that in fact D is itself a hyperplane.

   (c) Prove that the only holomorphic sections of $\mathcal{O}(1) \to \mathbb{CP}^n$ are those coming from the linear functions $\mathbb{C}^{n+1} \to \mathbb{C}$.

4. Prove that the canonical bundle of $\mathbb{CP}^n$ is isomorphic to $\mathcal{O}(-n-1)$.

   Hint: use the $n+1$ affine charts $\{z_j \neq 0\}$ to write down a meromorphic volume form on $\mathbb{CP}^n$ with a simple pole along each coordinate hyperplane $\{z_j = 0\}$.

5. Prove Corollary 2.25.

6. Let s be a holomorphic section of $\mathcal{O}(d) \to \mathbb{CP}^2$ for which $s^{-1}(0)$ is a smooth irreducible hypersurface Σ.

   (a) Prove that $[\Sigma] \cdot [\Sigma] = d^2$.

   (b) Prove that the genus of Σ is given by

   $$g(\Sigma) = \frac{(d-1)(d-2)}{2}$$

7. Let s be a holomorphic section of $\mathcal{O}(d) \to \mathbb{CP}^n$ for which $s^{-1}(0)$ is a smooth irreducible hypersurface X. Prove that the canonical bundle of X is trivial when $d = n+1$.